

# Some properties of sets in the plane closed under linear extrapolation by a fixed parameter

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## Abstract

Fix any  $\lambda \in \mathbb{C}$ . We say that a set  $S \subseteq \mathbb{C}$  is  $\lambda$ -convex if, whenever  $a$  and  $b$  are in  $S$ , the point  $(1 - \lambda)a + \lambda b$  is also in  $S$ . If  $S$  is also (topologically) closed, then we say that  $S$  is  $\lambda$ -clonvex. We investigate the properties of  $\lambda$ -convex and  $\lambda$ -clonvex sets and prove a number of facts about them. Letting  $R_\lambda \subseteq \mathbb{C}$  be the least  $\lambda$ -clonvex superset of  $\{0, 1\}$ , we show that if  $R_\lambda$  is convex in the usual sense, then  $R_\lambda$  must be either  $[0, 1]$  or  $\mathbb{R}$  or  $\mathbb{C}$ , depending on  $\lambda$ . We investigate which  $\lambda$  make  $R_\lambda$  convex, derive a number of conditions equivalent to  $R_\lambda$  being convex, give several conditions sufficient for  $R_\lambda$  to be convex or not convex (in particular,  $R_\lambda$  is either convex or discrete), and investigate the properties of some particular discrete  $R_\lambda$ , as well as other  $\lambda$ -convex sets.

Our work combines elementary concepts and techniques from algebra and plane geometry.

## 1 Introduction

**Definition 1.** Fix a number  $\lambda \in \mathbb{C}$  and a set  $S \subseteq \mathbb{C}$ .

1. We say that  $S$  is  $\lambda$ -convex iff for every  $a, b \in S$ , the point  $(1 - \lambda)a + \lambda b$  is in  $S$ .
2. We say that  $S$  is  $\lambda$ -convex closed (or  $\lambda$ -clonvex for short) iff  $S$  is  $\lambda$ -convex and (topologically) closed.

In either case, we say that  $S$  is *nontrivial* if  $S$  contains at least two distinct elements. We will informally say, “ $\lambda$ -c[l]onvex” when we want to assert analogous things about both notions, respectively.

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By the definition of convexity, a set  $S \subseteq \mathbb{C}$  is convex if and only if  $S$  is  $\lambda$ -convex for all  $\lambda \in ]0, 1[$ . Definition 1 above is in part motivated by the following additional observation (Proposition 13, below): If  $S$  is a *closed* set, then for *any fixed*  $\lambda \in ]0, 1[$ , we have that  $S$  is convex if and only if  $S$  is  $\lambda$ -convex. For this reason, we concentrate on the notion of  $\lambda$ -clonvexity, with the weaker notion of  $\lambda$ -convexity playing an ancillary role. We are generally interested in  $\lambda$ -clonvexity for  $\lambda \notin [0, 1]$ , and we are particularly interested in minimal nontrivial  $\lambda$ -clonvex sets.

**Definition 2.** For any  $\lambda \in \mathbb{C}$  and any set  $S \subseteq \mathbb{C}$ ,

1. We define the  $\lambda$ -convex closure of  $S$ , denoted  $Q_\lambda(S)$ , to be the  $\subseteq$ -minimum  $\lambda$ -convex superset of  $S$ . We let  $Q_\lambda$  be shorthand for  $Q_\lambda(\{0, 1\})$ , the  $\lambda$ -convex closure of  $\{0, 1\}$ .
2. We define the  $\lambda$ -clonvex closure of  $S$ , denoted  $R_\lambda(S)$ , to be the  $\subseteq$ -minimum  $\lambda$ -clonvex superset of  $S$ . We let  $R_\lambda$  be shorthand for  $R_\lambda(\{0, 1\})$ , the  $\lambda$ -clonvex closure of  $\{0, 1\}$ .

$R_\lambda$  is a minimal nontrivial  $\lambda$ -clonvex set because it is generated by just two distinct points. We choose the points 0 and 1 for convenience, but since  $\lambda$ -clonvexity is invariant under orientation-preserving similarity transformations (i.e.,  $\mathbb{C}$ -affine transformations, i.e., polynomials of degree 1; see Definition 3, below), any two initial points would yield a set with the same essential properties. Our main goal, then, is to characterize  $R_\lambda$  for as many  $\lambda$  as we can.

Despite an honest search of the literature, we so far cannot find any previous work related to this topic. Our work bears some superficial resemblance to dynamical systems in that an operation is applied repeatedly to some initial points yielding some asymptotic behavior. The Mandelbrot set and the various Julia sets (see [4], for example) are defined in this way. Our operation is binary, however, requiring *two* existing points to produce a new point, whereas the functions iterated in the study of dynamical systems are usually unary.

There may also be some connection between certain of our sets (for  $\lambda = 1 + \varphi = (3 + \sqrt{5})/2$ ) and Penrose tilings [5, 3], but we have yet to establish any direct connection.

## 2 Basics

We start with a few basic facts and definitions. In this paper, we call a theorem a “Fact” when it is either immediately obvious or has a routine, straightforward proof. We omit the proofs of Facts.

For  $z \in \mathbb{C}$ , we let  $\Re(z)$  and  $\Im(z)$  denote the real and imaginary parts of  $z$ , respectively, and we let  $z^*$  denote the complex conjugate of  $z$ .

Any topological references assume the usual topology on  $\mathbb{C} \cong \mathbb{R}^2$ . For  $A \subseteq \mathbb{C}$ , we let  $\overline{A}$  denote the topological closure of  $A$ .

We use the symbol  $:=$  to mean, “equals by definition.” We set  $\tau := 2\pi$  throughout. For any  $x \in \mathbb{R}$ , let  $x \bmod \tau$  denote the unique  $y \in [0, \tau[$  such that  $(x - y)/\tau$  is an integer.

We let  $\mathbb{Z}^+$  denote the set of positive integers.

Operations on numbers lift to operations on sets of numbers in the usual way.

**Definition 3.** For any  $a, b \in \mathbb{C}$ , define the function  $\rho_{a,b} : \mathbb{C} \rightarrow \mathbb{C}$  by

$$\rho_{a,b}(z) := (1 - z)a + zb$$

for all  $z \in \mathbb{C}$ . For fixed  $\lambda \in \mathbb{C}$ , we call  $\rho_{a,b}(\lambda)$  the  $\lambda$ -extrapolant of  $a$  and  $b$ , and we say that  $\rho_{a,b}(\lambda)$  is obtained from  $a$  and  $b$  by  $\lambda$ -extrapolation.

Then the first property in Definition 1 can be replaced with

1. for all  $a, b \in S$ , the point  $\rho_{a,b}(\lambda)$  is in  $S$ .

In short,  $S$  is closed under (linear)  $\lambda$ -extrapolation. Of course, if  $0 \leq \lambda \leq 1$ , then this might more appropriately be called  $\lambda$ -interpolation, but as we will see, the case where  $\lambda \notin [0, 1]$  is much more interesting.

**Fact 4.** For all  $a, b \in \mathbb{C}$ ,

1.  $\rho_{a,b}$  is the unique  $\mathbb{C}$ -affine map (polynomial of degree  $\leq 1$ ) that maps 0 to  $a$  and 1 to  $b$ .
2.  $\rho_{a,b}$  is continuous.
3. If  $a \neq b$ , then  $\rho_{a,b}$  is a bijection (a homeomorphism, in fact, and the unique orientation-preserving similarity transformation mapping 0 to  $a$  and 1 to  $b$ ), and for all  $z \in \mathbb{C}$ ,

$$(\rho_{a,b})^{-1}(z) = \frac{z - a}{b - a}.$$

It follows that  $(\rho_{a,b})^{-1} = \rho_{x,y}$ , where

$$x = \frac{a}{a - b}, \quad y = \frac{a - 1}{a - b}.$$

4. For all  $x, y \in \mathbb{C}$ ,

$$\rho_{a,b} \circ \rho_{x,y} = \rho_{\rho_{a,b}(x), \rho_{a,b}(y)}.$$

Equivalently, we have the following distributive law: for all  $z \in \mathbb{C}$ ,

$$\rho_{a,b}((1 - z)x + zy) = (1 - z)\rho_{a,b}(x) + z\rho_{a,b}(y).$$

**Lemma 5.** For any  $a, b, \lambda \in \mathbb{C}$  and  $S \subseteq \mathbb{C}$ , if  $S$  is  $\lambda$ -convex (respectively,  $\lambda$ -clonvex), then  $\rho_{a,b}(S)$  is  $\lambda$ -convex (respectively,  $\lambda$ -clonvex).

*Proof.* Suppose  $S$  is  $\lambda$ -convex. If  $a = b$ , then the statement is trivial, so we assume  $a \neq b$ . Fix any  $x, y \in \rho_{a,b}(S)$  and let  $u, v \in S$  be such that  $x = \rho_{a,b}(u)$  and  $y = \rho_{a,b}(v)$ . Then

$$\rho_{x,y}(\lambda) = \rho_{\rho_{a,b}(u), \rho_{a,b}(v)}(\lambda) = \rho_{a,b}(\rho_{u,v}(\lambda)).$$

We have  $\rho_{u,v}(\lambda) \in S$  because  $S$  is  $\lambda$ -convex; thus  $\rho_{x,y}(\lambda) \in \rho_{a,b}(S)$ . This proves that  $\rho_{a,b}(S)$  is  $\lambda$ -convex.

If, in addition,  $S$  is closed, then so is  $\rho_{a,b}(S)$ , because  $\rho_{a,b}$  is a homeomorphism. This proves that  $\rho_{a,b}$  preserves  $\lambda$ -clonvexity as well.  $\square$

**Fact 6.**  $Q_\lambda(S) \subseteq R_\lambda(S)$  for all  $\lambda \in \mathbb{C}$  and  $S \subseteq \mathbb{C}$ .

The next lemma gives a basic relationship between  $Q_\lambda$  and  $R_\lambda$ . Recall that  $\overline{A}$  denotes the topological closure of set  $A$ .

**Lemma 7.** For any  $\lambda \in \mathbb{C}$  and  $S \subseteq \mathbb{C}$ ,  $\overline{Q_\lambda(S)} = R_\lambda(S)$ .

*Proof.* The  $\subseteq$ -containment is obvious because  $R_\lambda(S)$  is closed and contains  $Q_\lambda(S)$ . For the  $\supseteq$ -containment, we just need to show that  $\overline{Q_\lambda(S)}$  is  $\lambda$ -convex. For any  $a, b \in \overline{Q_\lambda(S)}$ , let  $c := \rho_{a,b}(\lambda)$ . We show that  $c \in \overline{Q_\lambda(S)}$ . For any  $\varepsilon > 0$ , there exist points  $x, y \in Q_\lambda(S)$  such that  $|x - a| < \varepsilon/(2M)$  and  $|y - b| < \varepsilon/(2M)$ , where  $M := \max(|\lambda|, |1 - \lambda|)$ . Then  $z := \rho_{x,y}(\lambda)$  is in  $Q_\lambda(S)$ , and

$$|z - c| = |(1 - \lambda)(x - a) + \lambda(y - b)| \leq |1 - \lambda||x - a| + |\lambda||y - b| \leq M(|x - a| + |y - b|) < \varepsilon.$$

So there exists a point  $z \in Q_\lambda(S)$  within  $\varepsilon$  of  $c$ . Since  $\varepsilon$  was arbitrary, this shows that  $c \in \overline{Q_\lambda(S)}$ .  $\square$

The next lemma helps to justify our arbitrary choice of 0 and 1 in the definitions of  $Q_\lambda$  and  $R_\lambda$ .

**Lemma 8.** *For any  $a, b, \lambda \in \mathbb{C}$  and any set  $S \subseteq \mathbb{C}$ ,*

$$\begin{aligned}\rho_{a,b}(Q_\lambda(S)) &= Q_\lambda(\rho_{a,b}(S)) , \\ \rho_{a,b}(R_\lambda(S)) &= R_\lambda(\rho_{a,b}(S)) .\end{aligned}$$

*In particular,  $\rho_{a,b}(R_\lambda)$  is the  $\lambda$ -clonvex closure of  $\{a, b\}$ .*

*Proof.* If  $a = b$ , then the lemma is trivially true, so we assume  $a \neq b$ . For the first equality, the  $\supseteq$  part follows from two facts: (i)  $S \subseteq Q_\lambda(S)$ , and so  $\rho_{a,b}(S) \subseteq \rho_{a,b}(Q_\lambda(S))$ ; (ii)  $\rho_{a,b}(Q_\lambda(S))$  is  $\lambda$ -convex by Lemma 5. For the  $\subseteq$  part, choose  $x, y$  such that  $\rho_{x,y} = (\rho_{a,b})^{-1}$ , and set  $T := \rho_{a,b}(S)$ ; apply the  $\supseteq$  part we just proved to  $x, y$ , and  $T$  to get

$$\rho_{x,y}(Q_\lambda(T)) \supseteq Q_\lambda(\rho_{x,y}(T)) ,$$

then apply  $\rho_{a,b}$  to both sides to get

$$Q_\lambda(\rho_{a,b}(S)) = Q_\lambda(T) \supseteq \rho_{a,b}(Q_\lambda(\rho_{x,y}(T))) = \rho_{a,b}(Q_\lambda(S)) .$$

For the second equality, we have

$$\rho_{a,b}(R_\lambda(S)) = \rho_{a,b}(\overline{Q_\lambda(S)}) = \overline{\rho_{a,b}(Q_\lambda(S))} = \overline{Q_\lambda(\rho_{a,b}(S))} = R_\lambda(\rho_{a,b}(S)) .$$

The first and last equalities follow from Lemma 7; the second equality follows from the fact that  $\rho_{a,b}$  is a homeomorphism; we just proved the third equality.  $\square$

The next fact can be seen by noticing that  $\rho_{a,b}(\lambda) = \rho_{b,a}(1 - \lambda)$  for all  $a, b, \lambda \in \mathbb{C}$ .

**Fact 9.** *A set is  $\lambda$ -c[l]onvex if and only if it is  $(1 - \lambda)$ -c[l]onvex. Thus  $Q_\lambda(S) = Q_{1-\lambda}(S)$  and  $R_\lambda(S) = R_{1-\lambda}(S)$  for any  $S \subseteq \mathbb{C}$  and  $\lambda \in \mathbb{C}$ .*

**Fact 10.** *For any  $\lambda \in \mathbb{C}$ ,*

- $Q_\lambda(S)^* = Q_{\lambda^*}(S^*)$  and  $R_\lambda(S)^* = R_{\lambda^*}(S^*)$  for any  $S \subseteq \mathbb{C}$ .
- In particular,  $(Q_\lambda)^* = Q_{\lambda^*}$  and  $(R_\lambda)^* = R_{\lambda^*}$ .
- Thus  $R_\lambda$  is convex if and only if  $R_{\lambda^*}$  is convex.

The following geometric picture of  $a, b$  and  $\rho_{a,b}(\lambda)$  is especially useful for constructions involving  $\lambda \in \mathbb{C} - \mathbb{R}$ . See Figure 1.

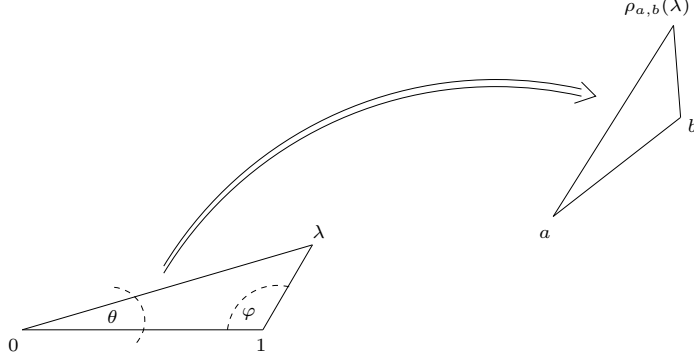


Figure 1: The affine transformation that takes  $(0, 1, \lambda) \mapsto (a, b, \rho_{a,b}(\lambda))$  preserves angles, so that the triangles  $(0, 1, \lambda)$  and  $(a, b, \rho_{a,b}(\lambda))$  are similar.

**Fact 11.** *By Fact 4.1, for any  $a, b, \lambda \in \mathbb{C}$ , the points  $a$ ,  $b$ , and  $\rho_{a,b}(\lambda)$  form a triangle that is similar to the one formed by  $0$ ,  $1$ , and  $\lambda$ .*

**Definition 12.** We call the angles  $\theta$  (formed by  $(\lambda, 0, 1)$ ) and  $\varphi$  (formed by  $(\lambda, 1, 0)$ ), as indicated in Figure 1, the *characteristic angles* of  $\lambda$ . We assume  $0 < \theta, \varphi < \pi$ .

**Proposition 13.** *If  $S \subseteq \mathbb{C}$  is closed, then for any fixed  $\lambda \in ]0, 1[$ , we have that  $S$  is convex if and only if  $S$  is  $\lambda$ -convex.*

*Proof.* Clearly, if  $S$  is convex, it is  $\lambda$ -convex for any fixed  $\lambda \in ]0, 1[$ .

Now suppose  $S$  is  $\lambda$ -convex for some fixed  $\lambda \in ]0, 1[$ . Let  $I := [0, 1]$ . Then consider two points  $a, b \in S$ , and the line segment  $L := \rho_{a,b}(I) = \{x \mid x = (1 - \ell)a + \ell b \text{ for some } \ell \in I\}$ , which connects  $a$  and  $b$ . We now show that  $S$  is dense in the set  $L$ . This is sufficient for the proposition: Since  $S$  is closed by hypothesis, this implies that in fact  $L \subseteq S$ , from which it follows that  $S$  is convex.

Suppose, then, that  $S$  is not dense in  $L$ . That is, there is some nonempty open<sup>1</sup> subset of  $L$  that does not intersect with  $S$ . Since  $\rho_{a,b}$  is continuous, this implies that there is some open interval  $K \subseteq I$  such that  $\rho_{a,b}(K) \cap S = \emptyset$ . Let  $J$  be the largest open interval in  $I$ , containing  $K$ , whose image under  $\rho_{a,b}$  does not intersect with  $S$ . More formally,

$$J = \cup \{K' \mid K \subseteq K' \subseteq I, K' \text{ is an open interval, and } \rho_{a,b}(K') \cap S = \emptyset\}.$$

Then there exists  $\lambda_1, \lambda_2 \in \mathbb{R}$  such that  $0 \leq \lambda_1 < \lambda_2 \leq 1$ ,  $\rho_{a,b}(\lambda_1), \rho_{a,b}(\lambda_2) \in S$ , and  $J = ]\lambda_1, \lambda_2[$ . Let  $p_1 = \rho_{a,b}(\lambda_1)$ ,  $p_2 = \rho_{a,b}(\lambda_2)$ , and  $x = \rho_{p_1, p_2}(\lambda) = (1 - \lambda)p_1 + \lambda p_2$ . A simple calculation shows that  $x = [(1 - \lambda)(1 - \lambda_1) + \lambda(1 - \lambda_2)]a + [(1 - \lambda)\lambda_1 + \lambda\lambda_2]b$ , and furthermore that  $\lambda_1 < (1 - \lambda)\lambda_1 + \lambda\lambda_2 < \lambda_2$ , which implies  $(1 - \lambda)\lambda_1 + \lambda\lambda_2 \in J$ . Since  $p_1, p_2 \in S$ , by construction,  $x \in S$  as well. But  $x = \rho_{a,b}((1 - \lambda)\lambda_1 + \lambda\lambda_2)$ , where  $(1 - \lambda)\lambda_1 + \lambda\lambda_2 \in J$ . Thus  $\rho_{a,b}(J) \cap S \neq \emptyset$ , which contradicts the fact that the image of  $J$  does not intersect with  $S$ .  $\square$

**Corollary 14.** *If  $T \subseteq \mathbb{C}$  and  $0 < \lambda < 1$ , then  $R_\lambda(T)$  is the closure of the convex hull of  $T$ .*

<sup>1</sup>with respect to the induced topology on  $L$

*Proof.* Let  $S := R_\lambda(T)$ , and let  $S'$  be the (topological) closure of the convex hull of  $T$ . We have  $T \subseteq S$  and  $S$  is closed and  $\lambda$ -convex, whence it follows that  $S$  is convex by Proposition 13, and thus  $S' \subseteq S$ . Conversely, the closure of the convex hull of any set is also convex. Thus  $S'$  is  $\lambda$ -convex by the same proposition, and this together with the inclusion  $T \subseteq S'$  imply  $S \subseteq S'$ .  $\square$

Now we consider  $R_\lambda = R_\lambda(\{0, 1\})$ . If  $R_\lambda$  happens to be convex, then characterizing  $R_\lambda$  is easy.

**Theorem 15.** *Suppose  $R_\lambda$  is convex.*

1. *If  $\lambda \in [0, 1]$ , then  $R_\lambda = [0, 1]$ .*
2. *If  $\lambda \in \mathbb{R} - [0, 1]$ , then  $R_\lambda = \mathbb{R}$ .*
3. *If  $\lambda \in \mathbb{C} - \mathbb{R}$ , then  $R_\lambda = \mathbb{C}$ .*

It will be convenient later to define the following:

**Definition 16.** *For any  $\lambda \in \mathbb{C}$ , define*

$$F_\lambda = \begin{cases} [0, 1] & \text{if } \lambda \in [0, 1], \\ \mathbb{R} & \text{if } \lambda \in \mathbb{R} - [0, 1], \\ \mathbb{C} & \text{if } \lambda \in \mathbb{C} - \mathbb{R}. \end{cases}$$

Then Theorem 15 states simply that if  $R_\lambda$  is convex, then  $R_\lambda = F_\lambda$ .

*Proof.* For (1), we have  $[0, 1] \subseteq R_\lambda$  by convexity, and if  $\lambda \in [0, 1]$ , then it is obvious that  $[0, 1]$  is  $\lambda$ -convex, since  $(1 - \lambda)a + \lambda b$  always lies on the line segment connecting  $a$  and  $b$ . Thus  $R_\lambda \subseteq [0, 1]$  by the minimality of  $R_\lambda$ .

For (2), we can assume WLOG that  $\lambda > 1$  (otherwise consider  $1 - \lambda$  and use Fact 9). Certainly,  $\lambda^0 = 1 \in R_\lambda$ , and if  $\lambda^n \in R_\lambda$  for some integer  $n \geq 0$ , then  $\lambda^{n+1} = \rho_{0, \lambda^n}(\lambda) \in R_\lambda$  as well. Thus by induction,  $\lambda^n \in R_\lambda$  for all integers  $n \geq 0$ . Since the sequence  $1, \lambda, \lambda^2, \lambda^3, \dots$  increases without bound, we have  $[1, \infty[ \subseteq R_\lambda$  by convexity. Similarly, the sequence  $1, 1 - \lambda, 1 - \lambda^2, 1 - \lambda^3, \dots$  lies entirely within  $R_\lambda$  (by induction, if  $1 - \lambda^n$  is in  $R_\lambda$ , then so is  $1 - \lambda^{n+1} = \rho_{1, 1 - \lambda^n}(\lambda)$ ). This latter sequence decreases without bound, and thus  $]-\infty, 1] \subseteq R_\lambda$  by convexity.

For (3), we use a trick suggested by George McNulty: we show that  $R_\lambda$  is open, and thus, since  $R_\lambda$  is nonempty and also closed, we must have  $R_\lambda = \mathbb{C}$ . Since  $\lambda \notin \mathbb{R}$ , we can represent  $\lambda$  in polar form as  $\lambda = re^{i\theta}$ , where  $r = |\lambda| > 0$ , and  $\theta = \arg \lambda \in \mathbb{R}$  is not a multiple of  $\pi$ . The value of  $\theta$  is determined modulo  $\tau$ , and so we take  $\theta$  to have the least possible absolute value, giving  $0 < |\theta| < \pi$ . Now consider any point  $a \in R_\lambda$ . Since  $R_\lambda$  has at least two points, there is some other point  $b \in R_\lambda - \{a\}$ . Now define the following sequence of points, all of which are in  $R_\lambda$ :

$$\begin{aligned} b_0 &:= b, \\ b_1 &:= \rho_{a, b_0}(\lambda), \\ &\vdots \\ b_{i+1} &:= \rho_{a, b_i}(\lambda), \\ &\vdots \end{aligned}$$

Set

$$k := \left\lfloor \frac{\pi}{|\theta|} \right\rfloor + 1 ,$$

the least integer such that  $k|\theta| > \pi$ . Then  $a$  lies in the interior of the convex hull of  $\{b_0, b_1, \dots, b_k\}$ , as illustrated in Figure 2.

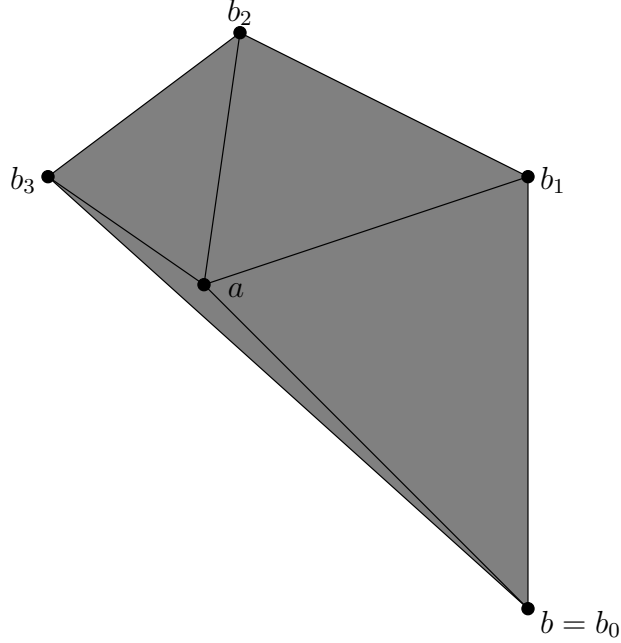


Figure 2: In this example where  $\lambda = (1 + 2i)/3$  and  $k = 3$ , the point  $a$  lies in the interior of the convex hull of  $\{b_0, b_1, b_2, b_3\}$ .

Since  $R_\lambda$  is convex, it contains this convex hull, whence  $a$  lies in the interior of  $R_\lambda$ . Since  $a \in R_\lambda$  was chosen arbitrarily, it follows that  $R_\lambda$  is open.  $\square$

In light of Theorem 15, most of the rest of the paper concentrates on determining, for various  $\lambda \in \mathbb{C}$ , whether or not  $R_\lambda$  is convex, and if not, characterizing  $R_\lambda$ . We start with a basic definition followed by a trivial observation.

**Definition 17.** Let  $\mathcal{C} := \{\lambda \in \mathbb{C} : R_\lambda \text{ is convex}\}$ .

**Fact 18.**

1.  $R_0 = R_1 = Q_0 = Q_1 = \{0, 1\}$ , hence  $0 \notin \mathcal{C}$  and  $1 \notin \mathcal{C}$ .
2. If  $0 < \lambda < 1$ , then  $R_\lambda = [0, 1]$  by Proposition 13 and Theorem 15, hence  $]0, 1[ \subseteq \mathcal{C}$ .

**Lemma 19.** For any  $a, b, \lambda \in \mathbb{C}$  and any  $S \subseteq \mathbb{C}$ ,

1. if  $\rho_{a,b}(S) \subseteq Q_\lambda(S)$ , then  $\rho_{a,b}(Q_\lambda(S)) \subseteq Q_\lambda(S)$ ;
2. if  $\rho_{a,b}(S) \subseteq R_\lambda(S)$ , then  $\rho_{a,b}(R_\lambda(S)) \subseteq R_\lambda(S)$ .

*Proof.* Using the first assumption and Lemma 8, we get

$$\rho_{a,b}(Q_\lambda(S)) = Q_\lambda(\rho_{a,b}(S)) \subseteq Q_\lambda(Q_\lambda(S)) = Q_\lambda(S) .$$

Using the second assumption and Lemma 8, we get

$$\rho_{a,b}(R_\lambda(S)) = R_\lambda(\rho_{a,b}(S)) \subseteq R_\lambda(R_\lambda(S)) = R_\lambda(S) .$$

□

Lemma 19 and Lemma 21 (below) have some useful corollaries.

**Corollary 20.** *For any  $a, b, \lambda \in \mathbb{C}$ ,*

1. *if  $a \in Q_\lambda$  and  $b \in Q_\lambda$ , then  $\rho_{a,b}(Q_\lambda) \subseteq Q_\lambda$ ;*
2. *if  $a \in R_\lambda$  and  $b \in R_\lambda$ , then  $\rho_{a,b}(R_\lambda) \subseteq R_\lambda$ .*

*Proof.* Set  $S = \{0, 1\}$  and use Lemma 19. □

Part (1.) of the next lemma will be used in Section 13.

**Lemma 21.** *For any  $\lambda, \mu \in \mathbb{C}$  and  $S \subseteq \mathbb{C}$ ,*

1. *if  $\mu \in Q_\lambda$ , then  $Q_\lambda(S)$  is  $\mu$ -convex, and consequently,  $Q_\mu(S) \subseteq Q_\lambda(S)$ ;*
2. *if  $\mu \in R_\lambda$ , then  $R_\lambda(S)$  is  $\mu$ -clonvex, and consequently,  $R_\mu(S) \subseteq R_\lambda(S)$ .*

*Proof.* For part (1.), suppose  $\mu \in Q_\lambda$ . Then for any  $a, b \in Q_\lambda(S)$ ,

$$\rho_{a,b}(\mu) \in \rho_{a,b}(Q_\lambda) = Q_\lambda(\{a, b\}) \subseteq Q_\lambda(Q_\lambda(S)) = Q_\lambda(S) ,$$

where the equation follows from Lemma 8 with  $S = \{0, 1\}$ . This shows that  $Q_\lambda(S)$  is  $\mu$ -convex. A similar argument holds for part (2.). □

**Corollary 22.** *For any  $\lambda, \mu \in \mathbb{C}$ ,*

1. *if  $\mu \in Q_\lambda$ , then  $Q_\lambda$  is  $\mu$ -convex, and consequently,  $Q_\mu \subseteq Q_\lambda$ ;*
2. *if  $\mu \in R_\lambda$ , then  $R_\lambda$  is  $\mu$ -clonvex, and consequently,  $R_\mu \subseteq R_\lambda$ .*

**Definition 23.** For any  $S \subseteq \mathbb{C}$ , define  $1 - S := \{1 - x \mid x \in S\}$ .

Note that  $1 - S = \rho_{1,0}(S)$ , for any  $S \subseteq \mathbb{C}$ .

**Corollary 24.**  $R_\lambda = 1 - R_\lambda$  for any  $\lambda \in \mathbb{C}$ .

*Proof.* We have

$$1 - R_\lambda = \rho_{1,0}(R_\lambda) \subseteq R_\lambda = \rho_{1,0}(\rho_{1,0}(R_\lambda)) \subseteq \rho_{1,0}(R_\lambda) = 1 - R_\lambda .$$

Both  $\subseteq$ -steps follow from Corollary 20. □

**Corollary 25.** *For any  $\lambda, \mu \in \mathbb{C}$ , if  $\mu \in R_\lambda$  and  $R_\mu$  is convex, then  $R_\lambda$  is convex.*



*Proof.* Assume  $\mu \in R_\lambda$  and  $R_\mu$  is convex. To show that  $R_\lambda$  is convex, it suffices to show that for any  $a, b \in R_\lambda$  and  $x \in [0, 1]$ , the point  $\rho_{a,b}(x)$  is in  $R_\lambda$ . We have  $0, 1 \in R_\mu$ , and so by Corollary 22 and the convexity of  $R_\mu$ , we have

$$[0, 1] \subseteq R_\mu \subseteq R_\lambda .$$

Thus, for any  $x \in [0, 1]$ , we have

$$\rho_{a,b}(x) \in \rho_{a,b}([0, 1]) \subseteq \rho_{a,b}(R_\lambda) \subseteq R_\lambda .$$

□

**Corollary 26.** *For any  $\lambda \in \mathbb{C}$ ,  $\lambda \in \mathcal{C}$  if and only if  $R_\lambda \cap \mathcal{C} \neq \emptyset$ .*

**Proposition 27.** *If  $R_\lambda$  is convex, then all  $\lambda$ -clonvex sets are convex.*

*Proof.* Suppose  $R_\lambda$  is convex, and let  $A$  be any  $\lambda$ -clonvex set. For any  $a, b \in A$ , the line segment connecting  $a$  and  $b$  is  $\rho_{a,b}([0, 1])$ . Since  $R_\lambda$  is convex, we have  $[0, 1] \subseteq R_\lambda$ , and thus

$$\rho_{a,b}([0, 1]) \subseteq \rho_{a,b}(R_\lambda) = R_\lambda(\rho_{a,b}(\{0, 1\})) = R_\lambda(\{a, b\}) \subseteq R_\lambda(A) = A .$$

The first equality follows from Lemma 8; the last equality holds because  $A$  is  $\lambda$ -clonvex. □

### 3 Equivalent characterizations of convexity for $\lambda$ -clonvex sets

For any  $a, b \in \mathbb{C}$ , a *path* from  $a$  to  $b$  is a continuous function  $\sigma : [0, 1] \rightarrow \mathbb{C}$  such that  $\sigma(0) = a$  and  $\sigma(1) = b$ . A set  $S \subseteq \mathbb{C}$  is said to be *path-connected* if it contains a path between any two of its points.<sup>2</sup>

In this section we consider five possible properties of a  $\lambda$ -clonvex set and the implications between them. Throughout this section, we will adopt the convention that  $\lambda$  denotes an arbitrary complex number and that  $A$  denotes an arbitrary  $\lambda$ -clonvex set containing at least two distinct points. Here are the five properties we will consider:

1.  $A$  is convex.
2.  $A$  is path-connected.
3.  $A$  contains a path.
4.  $A$  has an accumulation point.
5. There exist  $a, b \in \mathbb{C}$  such that  $0 < |a - b| < 1$  and  $\rho_{a,b}(A) \subseteq A$  (i.e.,  $A$  is self-similar).

In particular, we show (Corollary 32, below) that these five properties are all equivalent when  $A = R_\lambda$ , while some implications do not hold for all  $\lambda$ -clonvex sets.

We refer to the above properties by their numbers in parentheses.

**Fact 28.** *For all  $\lambda$  and  $A$  subject to the convention above,  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ .*

---

<sup>2</sup>Strictly speaking, as we identify the path with the function  $\sigma$ , it is more accurate to say that  $S$  contains all the points in the image of some path connecting the two points. However, we will assume that the meaning will be clear from the context.

**Theorem 29.** *For all  $\lambda$  and  $A$  subject to the convention above, (1)  $\Rightarrow$  (5).*

*Proof.* Choose any point  $x \in A$ , and consider the map

$$\psi := \rho_{x,x+1} \circ \rho_{0,1/2} \circ (\rho_{x,x+1})^{-1}.$$

It is easy to check that for any  $z \in \mathbb{C}$ ,  $\psi(z)$  is the midpoint  $(x+z)/2$  of  $x$  and  $z$ . Thus  $\psi(A) \subseteq A$ , because  $A$  is assumed to be convex. Using Fact 4, we get  $\psi = \rho_{a,b}$ , where

$$a = \psi(0) = \frac{x}{2}, \quad b = \psi(1) = \frac{x}{2} + \frac{1}{2}.$$

We have  $|a - b| = 1/2$ , which implies (5).  $\square$

**Theorem 30.** *For all  $\lambda$  and  $A$  subject to the convention above, (5)  $\Rightarrow$  (4).*

*Proof.* Let  $a, b$  be such that  $0 < |a - b| < 1$  and  $\rho_{a,b}(A) \subseteq A$ . Letting  $\delta := |a - b|$ , we see that for any  $x, y \in \mathbb{C}$ ,

$$\rho_{a,b}(x) - \rho_{a,b}(y) = (1 - x)a + xb - (1 - y)a - yb = (b - a)(x - y),$$

and thus

$$|\rho_{a,b}(x) - \rho_{a,b}(y)| = \delta |x - y|. \tag{1}$$

It is easy to check that the map  $\rho_{a,b}$  on  $\mathbb{C}$  has the unique fixed point

$$z := \frac{a}{1 + a - b},$$

and a routine induction on  $n$  using Equation (1) shows that for any  $w \in \mathbb{C}$ ,  $|z - \rho_{a,b}^{(n)}(w)| = \delta^n |z - w|$  for  $n = 0, 1, 2, \dots$ . Thus if  $z \neq w$ , then  $z$  is an accumulation point of the sequence

$$w, \rho_{a,b}(w), \rho_{a,b}(\rho_{a,b}(w)), \dots, \rho_{a,b}^{(n)}(w), \dots$$

If, in addition,  $w \in A$  (and there must exist such a  $w$ , because  $A$  contains at least two points by convention), then all the elements of this sequence are in  $A$ . This fact is proved by induction on  $n$ : for the inductive step, we have that if  $\rho_{a,b}^{(n)}(w) \in A$  for some  $n \geq 0$ , then

$$\rho_{a,b}^{(n+1)}(w) = \rho_{a,b}(\rho_{a,b}^{(n)}(w)) \in \rho_{a,b}(A) \subseteq A,$$

the last containment following from the assumption. We then get  $z \in A$  by the closure of  $A$ .  $\square$

Recall the definition of  $F_\lambda$  in Definition 16.

**Theorem 31.** *For all  $\lambda$  and  $A$  subject to the convention above and such that  $\lambda \notin \{0, 1\}$  and  $A \subseteq F_\lambda$ , (4)  $\Rightarrow$  (1).*

*Proof.* We consider the case where  $\lambda \in \mathbb{R}$  first. This case is required, but it also gives a simpler version of the proof for when  $\lambda$  is complex. By Proposition 13, if  $0 < \lambda < 1$ , then  $A$  is convex, regardless of whether or not  $A \subseteq F_\lambda$  (which equals  $[0, 1]$  in this case). Therefore—since  $\lambda \notin \{0, 1\}$  by assumption—we may assume that  $\lambda \notin [0, 1]$ , in which case,  $A \subseteq F_\lambda = \mathbb{R}$ . Since  $\lambda$ -clonvexity is the same as  $(1 - \lambda)$ -clonvexity by Fact 9, we may further assume that  $\lambda > 1$ .

Now let  $C$  be the set of all accumulation points of  $A$ , and suppose that  $C \neq \emptyset$ . Note that  $C \subseteq A$ , because  $A$  is closed. We show for any  $x \in \mathbb{R}$ , that  $x \in C$ , from which it follows that  $A = \mathbb{R}$ . Let  $a \in C$  be closest to  $x$  among all the elements of  $C$ . Such a point  $a$  exists, because  $C$  is closed and nonempty. If  $x = a$ , then we are done, so suppose that  $x < a$  (there is no essential difference with the case in which  $a < x$ ). Then for some sequence  $\{a_n\} \subseteq A - \{a\}$ ,  $a = \lim_{n \rightarrow \infty} a_n$ . Define the sequence  $\{b_n\}$  as follows:

$$b_n = \begin{cases} \rho_{a, a_n}(\lambda) & \text{if } a_n < a, \\ \rho_{a_n, a}(\lambda) & \text{if } a_n > a. \end{cases}$$

It is easy to see that for all  $n$ ,  $b_n < a$ , and that  $\lim_{n \rightarrow \infty} b_n = a$ . Furthermore, for each  $n$ , we have  $b_n \in C$ , because

$$b_n = \begin{cases} \lim_{m \rightarrow \infty} \rho_{a_m, a_n}(\lambda) & \text{if } a_n < a, \\ \lim_{m \rightarrow \infty} \rho_{a_n, a_m}(\lambda) & \text{if } a_n > a, \end{cases}$$

and  $\rho_{a_m, a_n}(\lambda) \in A$  for all  $m, n$ . Since  $b_n$  converges to  $a$ , we find for sufficiently large  $n$  that  $x < b_n < a$ , so  $b_n$  is in  $C$  and is closer to  $x$  than  $a$  is, contradicting the hypothesis that  $a$  was the closest.

Now suppose that  $\lambda \notin \mathbb{R}$ , and that  $A$  is arbitrary but contains an accumulation point  $a$ . We show that in such a case,  $A = \mathbb{C}$ . We do this by showing that any point  $x \in \mathbb{C}$  is an accumulation point of  $A$ , from which the result follows by the closure of  $A$ . The proof is the same in spirit as for  $\lambda \in \mathbb{R}$ , but here there is no division into cases as to whether  $x$  is to the “left” or “right” of  $a$ . For ease of illustration, we will assume that  $\Im(\lambda) > 0$ , as depicted in Fig. 1. The case where  $\Im(\lambda) < 0$  is entirely similar.

The proof is by contradiction. Again, let  $C \subseteq A$  be the set of all accumulation points of  $A$ , and let  $b \in C$  be the closest to  $x$  of any point in  $C$ . Such a point  $b$  exists, because the set  $\{z \in C : |z - x| \leq |a - x|\}$ , as well as being nonempty, is closed and bounded, and hence compact. Now assume for the sake of contradiction that  $x \notin A$ . Then  $x \neq b$ . Draw a circle with  $x$  as the center and  $b$  on the circle. Let  $D$  denote the open disk bounded by the circle. We will show now that there exists a point in  $C \cap D$ , which contradicts the hypothesis that  $b$  is the closest point in  $C$  to  $x$ . For ease in visualization, suppose  $b$  is at the top of the circle (see Fig. 3).

Suppose the sequence  $\{b_n\} \subseteq A - \{b\}$  converges to  $b$ , that is,  $\lim_{n \rightarrow \infty} b_n = b$ . Let  $\{c_n\}$  denote the sequence defined by  $c_n = \rho_{b_n, b}(\lambda)$  for all  $n$ . Note that for each  $n$ ,  $c_n \neq b$ , and moreover,  $c_n$  is itself in  $C$ , because

$$\lim_{m \rightarrow \infty} \rho_{b_n, b_m}(\lambda) = \lim_{m \rightarrow \infty} ((1 - \lambda)b_n + \lambda b_m) = (1 - \lambda)b_n + \lambda b = c_n.$$

Thus if any  $c_n \in D$ , then we are done, but it is possible that  $c_n \notin D$  for all  $n$ . We therefore show how to “rotate” the sequence  $\{c_n\}$  so that it is contained in the disk for all sufficiently large  $n$ . Let  $T$  denote the tangent to the circle at  $b$ . With  $\theta$  and  $\varphi$  denoting the characteristic angles of  $\lambda$  (see Definition 12 and Fig. 1), let  $\eta$  be any angle obeying  $0 < \eta < \min(\theta, \varphi, (\pi - \theta - \varphi)/2)$ . Form the

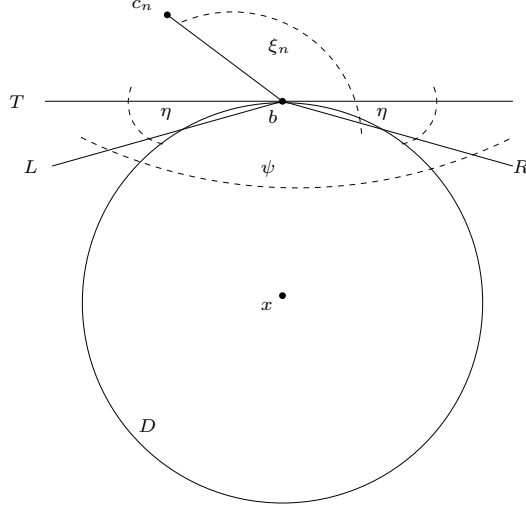


Figure 3: Theorem 31: Construction of accumulation point  $x$ .

two rays  $L$  and  $R$  intersecting at  $b$  and making an angle  $\eta$  with  $T$  as shown in the figure. Let  $\xi_n$  denote the angle formed by  $R$  and the line segment  $(b, c_n)$  connecting  $c_n$  with  $b$ , also as shown in the figure. Since  $c_n$  could be anywhere except  $b$ , we have  $0 \leq \xi_n < \tau$ , where  $\xi_n = 0$  corresponds to the ray  $R$ . Let  $\psi$  denote the angle subtended by  $L$  and  $R$ ; thus  $\psi = \pi - 2\eta$ . Note that by the choice of  $\eta$ , we have  $\varphi < \theta + \varphi < \pi - 2\eta = \psi$ . Our goal now is to find an accumulation point below the lines  $L$  and  $R$ , and inside  $D$ .

To do this, let  $q_n$  denote the least integer such that  $q_n\varphi > \xi_n$ . Then  $q_n\varphi = (q_n - 1)\varphi + \varphi \leq \xi_n + \varphi < \xi_n + \psi$ . Thus  $\xi_n < q_n\varphi < \xi_n + \psi$ , so that the angle  $q_n\varphi$  takes us from the line segment  $(b, c_n)$  *clockwise* to a ray through  $b$ , below  $T$ , and strictly between  $L$  and  $R$ . Note that since  $\xi_n < \tau$ ,  $q_n - 1 \leq \lfloor \tau/\varphi \rfloor$ , so for all  $n$ ,  $q_n$  can only take on a finite number of values independent of  $n$ .

Next, for each  $n$ , form the finite sequence  $c_n^{(0)}, c_n^{(1)}, \dots, c_n^{(q_n)}$ , where

$$\begin{aligned} c_n^{(0)} &= c_n \\ c_n^{(1)} &= \rho_{c_n^{(0)}, b}(\lambda) \\ &\vdots \\ c_n^{(i+1)} &= \rho_{c_n^{(i)}, b}(\lambda) \\ &\vdots \end{aligned}$$

(This is essentially the same construction as in Theorem 15. Also note Figure 2, although it is not necessary here to form a convex hull.) Each  $c_n^{(i)}$  is in  $A$ , similarly to  $c_n$ . By the definition of  $\varphi$ , for each  $i$ , the angle between the line segment  $(b, c_n^{(i)})$  and  $(b, c_n^{(i+1)})$  is  $\varphi$ . Thus the angle between line segments  $(b, c_n)$  and  $(b, c_n^{(q_n)})$  is  $q_n\varphi$ . Hence, the point  $c_n^{(q_n)}$  is in the desired region beneath  $L$  and  $R$ . However, it may be too far from  $b$  to be in the interior of  $D$ . Now observe that, by virtue of the fact that  $c_n^{(i+1)}$  is always constructed from  $b$  and  $c_n^{(i)}$  via similar triangles, there is a constant  $k$  such that  $|b - c_n^{(i+1)}| \leq k|b - c_n^{(i)}|$ . Thus  $|b - c_n^{(q_n)}| \leq k^{q_n}|b - c_n|$ . But since  $\{c_n\}$  converges to  $b$ , for

any  $\epsilon$ , there exists an  $n_0$  such that  $|b - c_{n_0}| \leq \epsilon/k^{\lfloor 2\pi/\varphi \rfloor + 1} \leq \epsilon/k^{q_{n_0}}$ . In that case,  $|b - c_{n_0}^{(q_{n_0})}| \leq \epsilon$ , so  $\epsilon$  may be chosen sufficiently small that  $c_{n_0}^{(q_{n_0})}$  is contained in  $D$ . (And indeed, the sequence  $\{c_n^{(q_n)}\}$  converges to  $b$ .)  $\square$

Some kind of constraint on  $\lambda$  and  $A$  in Theorem 31 is necessary to obtain the implication (4)  $\Rightarrow$  (1). For example, if  $\lambda \in \{0, 1\}$ , then any closed subset of  $\mathbb{C}$  is  $\lambda$ -convex, and so we may take  $A := \{0\} \cup \{1/n : n \in \mathbb{Z}^+\}$ , which has 0 as an accumulation point but is not convex. If  $\lambda \in \mathbb{R} - [0, 1]$  but  $A \not\subseteq \mathbb{R}$ , then the implication still holds provided either  $A$  lies entirely on a single line or  $A$  contains a nonempty open set (cf. Proposition 33, below). Otherwise, the implication may not hold: let  $\lambda := 2$  and consider the set  $A := R_2(\{0, 1, \sqrt{2}, i\})$ . Then it is a short exercise to show that

$$A = \{x + yi \mid x \in \mathbb{R} \text{ \& \; } y \in \mathbb{Z}\},$$

which has accumulation points (paths, in fact) but is not convex.

Property (5) of the next corollary provides a useful shortcut for proving that  $R_\lambda$  is convex.

**Corollary 32.** *For any  $\lambda \in \mathbb{C}$ , the following are equivalent:*

1.  $R_\lambda$  is convex.
2.  $R_\lambda$  is path-connected.
3.  $R_\lambda$  contains a path.
4.  $R_\lambda$  has an accumulation point.
5. There exist  $a, b \in R_\lambda$  such that  $0 < |a - b| < 1$ .

*Proof.* If  $A = R_\lambda$ , then we merely note that the property (5) of Corollary 32 is equivalent to the property (5) given earlier in this section: if  $a, b \in R_\lambda$ , then  $\rho_{a,b}(R_\lambda) \subseteq R_\lambda$  by Corollary 20. Conversely, if  $\rho_{a,b}(R_\lambda) \subseteq R_\lambda$ , then  $\{a, b\} = \rho_{a,b}(\{0, 1\}) \subseteq \rho_{a,b}(R_\lambda) \subseteq R_\lambda$ .  $\square$

We end this section with some basic facts about  $Q_\lambda(S)$  for certain  $\lambda$  and  $S$ . First we show that, given any disk  $D \subseteq \mathbb{C}$  and any  $\lambda \in \mathbb{C} - [0, 1]$ , we can construct a larger concentric disk  $D' \subseteq Q_\lambda(D)$  (analogous to the construction of successively larger intervals in the case  $\lambda \in \mathbb{R}$ ). From this it follows immediately that  $Q_\lambda(D) = \mathbb{C}$  (Proposition 33, below). For the moment, we will assume that  $\Im(\lambda) > 0$ . The case where  $\Im(\lambda) < 0$  is similar, and the case where  $\lambda \in \mathbb{R}$  is obvious.

First observe the elementary fact that any triangle with a given base has the same area as some isosceles triangle of the same height and base (see Figure 4). The angle  $\varphi$  between the equal sides is uniquely determined by the original triangle. In particular, for any  $\lambda$  with characteristic angle  $\theta$ , the angle  $\varphi$  is uniquely determined.

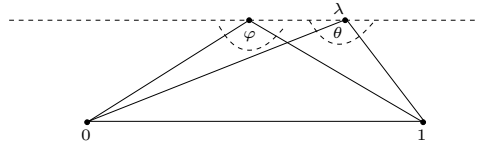


Figure 4: Proposition 33: Isosceles triangle of equal area.

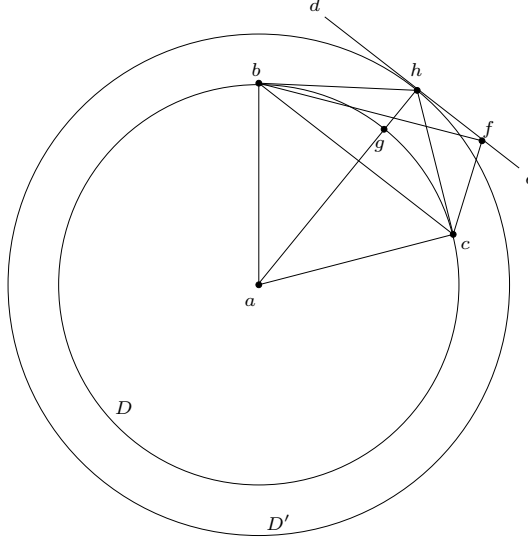


Figure 5: Proposition 33: Constructing a bigger disk.

Begin with a closed disk  $D$  centered at  $a$ , as depicted in Figure 5. Draw lines  $bh$  and  $hc$ , both tangent to the boundary of  $D$ , such that the angle  $bhc$  is the angle  $\varphi$  described above. In the figure, with the line  $de$  parallel to  $bc$ , the angle  $bfc$  can be taken to be the characteristic angle of  $\lambda$ . Hence the point  $f = \rho_{b,c}(\lambda)$  is contained in  $Q_\lambda(D)$ . Indeed, since the line segment  $bc$  is contained in  $D$ , the entire triangle  $bfc$  and its interior are contained in  $Q_\lambda(D)$  (each point  $x$  in this region equals  $\rho_{y,z}(\lambda)$  for points  $y, z$  on the line segment  $bc$ ). Hence the locus of all triangles of the form  $bfc$ , as the point  $h$  encircles the center  $a$ , is an annulus centered at  $a$ , which is all contained in  $Q_\lambda(D)$ . A lower bound on the area of this annulus can be obtained by computing the area of the smaller annulus swept out by the triangle  $bhc$ . Let  $r$  denote the radius of  $D$  (i.e., the length of the line  $ag$ ), and  $r'$  the radius of  $D'$  (i.e., the length of the line  $ah$ ). Then it is easy to see that  $r' = r / \sin(\varphi/2)$ . The area of the annulus  $D - D'$  is then  $\pi r^2((1/\sin(\varphi/2))^2 - 1)$ , a constant fraction of the area of  $D$ . We have thus found a larger concentric disk  $D' \supseteq D$ , with area proportional to the area of  $D$ , and contained in  $Q_\lambda(D)$ . Since this can be carried on indefinitely,  $\mathbb{C} \subseteq Q_\lambda(D)$ .

**Remark.** If we want to expand  $D$  to  $D'$  optimally in the construction above, then we choose  $b$  and  $c$  so that angle  $bac = \pi - \theta$  rather than  $\pi - \varphi$ , giving a ratio of  $|\lambda| + |1 - \lambda|$  in the radius of  $D'$  to that of  $D$ . This is true whether or not  $\lambda \in \mathbb{R}$ .  $\square$

Thus we have the following:

**Proposition 33.** Fix any  $\lambda \in \mathbb{C} - [0, 1]$  and  $A \subseteq \mathbb{C}$ .

1. If  $A$  includes a nonempty open subset of  $\mathbb{C}$ , then  $Q_\lambda(A) = R_\lambda(A) = \mathbb{C}$ .
2. If  $A$  includes a nonempty open subset of  $\mathbb{R}$ , then  $F_\lambda \subseteq Q_\lambda(A)$ .

*Proof.* Part (1.) follows from the discussion above. For Part (2.), we have two cases: (i)  $\lambda \in \mathbb{R}$  and  $F_\lambda = \mathbb{R}$ ; and (ii)  $\lambda \notin \mathbb{R}$  and  $F_\lambda = \mathbb{C}$ . In case (i), we apply a one-dimensional version of

the disk expansion construction above to expand any interval  $[a, b] \subseteq Q_\lambda(A)$  to a larger interval  $[c, d] \subseteq Q_\lambda(A)$ , where (assuming  $\lambda > 1$  without loss of generality)  $c = \rho_{b,a}(\lambda)$  and  $d = \rho_{a,b}(\lambda)$ . Note that for every point  $z \in [c, d]$  there exist  $x, y \in [a, b]$  such that  $z = \rho_{x,y}(\lambda)$ . The expansion is by a factor of  $2\lambda - 1 > 1$ . Applying the expansion repeatedly puts all of  $\mathbb{R}$  into  $Q_\lambda(A)$ . For case (ii), if we start with some interval  $[a, b] \subseteq A$ , then the entire triangle formed by  $a$ ,  $b$ , and  $\rho_{a,b}(\lambda)$  and its interior lies in  $Q_\lambda(A)$ . Indeed, for any point  $z$  inside this triangle, there exist  $x, y \in [a, b]$  such that  $z = \rho_{x,y}(\lambda)$ , as shown in Figure 6. Then applying Part (1.) to  $Q_\lambda(A)$  gives  $Q_\lambda(Q_\lambda(A)) = Q_\lambda(A) = \mathbb{C}$ .  $\square$

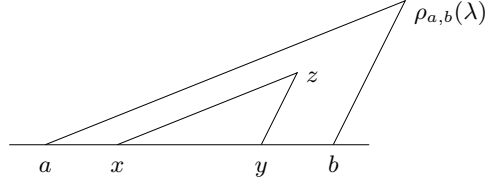


Figure 6: From the interval  $[a, b] \subseteq A$  we get a triangle in  $Q_\lambda(A)$  with a nonempty interior.

The proof above can easily be generalized to show that if  $A$  includes a differentiable path in  $\mathbb{C}$ , then  $Q_\lambda(A) = \mathbb{C}$  for all  $\lambda \in \mathbb{C} - \mathbb{R}$ . In fact, we can prove something much stronger:

**Theorem 34.** *If  $d$  is any path in  $\mathbb{C}$  that is not contained in a straight line, then  $Q_\lambda(d) = R_\lambda(d) = \mathbb{C}$  for all  $\lambda \in \mathbb{C} - [0, 1]$ .*

In Theorem 34, we do not require  $d$  to be differentiable, or even simple. We only require that  $d$  be continuous. We defer the proof of this theorem until Section 13.

## 4 Finding $\lambda$ such that $R_\lambda$ is convex

Corollary 32 itself has two useful corollaries:

**Corollary 35.**  *$R_\lambda$  is convex for any  $\lambda \in \mathbb{C}$  such that either  $0 < |\lambda| < 1$  or  $0 < |1 - \lambda| < 1$ .*

**Corollary 36.** *For any  $\lambda \in \mathbb{C}$ ,  $R_\lambda$  is convex if and only if there exists  $\mu \in R_\lambda$  such that either  $0 < |\mu| < 1$  or  $0 < |1 - \mu| < 1$ .*

*Proof.* The forward implication is obvious, since  $[0, 1] \subseteq R_\lambda$  if  $R_\lambda$  is convex. For the converse, we have  $R_\mu$  is convex by Corollary 35, whence  $R_\lambda$  is convex by Corollary 25.  $\square$

The proof of Corollary 32 is the last place where we explicitly use the fact that  $R_\lambda$  is closed. In fact, all convexity arguments from now on can follow directly or indirectly from Corollaries 25 and 36, or alternatively from Corollary 32. They now allow us to expand our set of  $\lambda$  such that  $R_\lambda$  is known to be convex.

**Proposition 37.** *If  $|\lambda| = 1$  and  $\lambda$  is neither a fourth nor a sixth root of unity, then  $R_\lambda$  is convex.*

*Proof.* Let  $\lambda \in \mathbb{C}$  be any point on the unit circle. Write  $\lambda = x + iy$  for real  $x, y$  such that  $x^2 + y^2 = 1$ . The following point is evidently in  $R_\lambda$ :

$$\mu := \rho_{1,\lambda}(\lambda) = \lambda^2 - \lambda + 1 = (x^2 - y^2 - x + 1) + (2xy - y)i = (2x^2 - x) + y(2x - 1)i = (2x - 1)\lambda.$$

Thus  $|\mu| = |2x - 1|$ . If  $0 < x < 1$ , then  $|\mu| < 1$ , and moreover,  $0 < |\mu|$  if  $x \neq 1/2$ . Corollary 35 then implies that  $R_\mu$  is convex (and thus  $R_\lambda$  is convex) for all  $x \in ]0, 1[ - \{1/2\}$ , which proves the current proposition when  $x \geq 0$ . (The cases where  $x \in \{0, 1/2, 1\}$  correspond to  $\lambda$  being a fourth or a sixth root of unity.)

Now assume  $x < 0$ . It is easy to check (on geometric grounds alone) that either  $\lambda^2$  or  $\lambda^3$  has positive real part, and so, provided  $\lambda^2$  (respectively  $\lambda^3$ ) is not a sixth root of unity, we have  $R_{\lambda^2}$  (respectively  $R_{\lambda^3}$ ) is convex by the argument in previous paragraph, and hence  $R_\lambda$  is convex. Thus the only cases left to show are where: (1)  $\lambda$  is neither a fourth nor a sixth root of unity; but (2)  $\lambda^2$  has nonpositive real part or is a sixth root of unity, and (3) similarly for  $\lambda^3$ . There are only four such cases:  $\lambda = e^{\pm i\tau(5/12)}$  and  $\lambda = e^{\pm i\tau(5/18)}$ . If  $\lambda = e^{\pm i\tau(5/12)}$ , then  $\lambda^5 = e^{\pm i\tau/12}$ , which has positive real part and is not a sixth root of unity. If  $\lambda = e^{\pm i\tau(5/18)}$ , then the same can be said for  $\lambda^4 = e^{\pm i\tau/9}$ . So we can apply the first paragraph argument to  $R_{\lambda^5}$  and  $R_{\lambda^4}$ , respectively.  $\square$

The converse of Proposition 37 for  $\lambda$  on the unit circle follows from the following fact:

**Fact 38.** *If  $D$  is any subring of  $\mathbb{C}$  that is (topologically) closed, and  $\lambda \in D$ , then  $R_\lambda \subseteq D$ .*

So in particular, if  $\lambda \in \mathbb{Z}$ , then  $R_\lambda \subseteq \mathbb{Z}$ ; if  $\lambda$  is a Gaussian integer, then  $R_\lambda$  consists only of Gaussian integers; if  $\lambda$  is an Eisenstein integer,<sup>3</sup> then  $R_\lambda$  consists only of Eisenstein integers. The fourth roots of unity are all Gaussian integers, and the sixth roots of unity are all Eisenstein integers. None of these choices of  $\lambda$  makes  $R_\lambda$  convex.

$R_\lambda$  is usually a proper subset of  $D$  for the choices of  $D$  mentioned above. More on this in Section 9.

**Corollary 39.** *If  $\lambda \in \mathbb{C} - [0, 1]$ , then  $R_\lambda$  is unbounded (and thus  $Q_\lambda$  is unbounded).*

*Proof.* Suppose  $\lambda \notin [0, 1]$ . If  $|\lambda| > 1$ , then  $R_\lambda$  is unbounded, since  $\lambda^n \in R_\lambda$  for all integers  $n > 0$ . Similarly, if  $|1 - \lambda| > 1$ , then  $R_\lambda$  is unbounded, since  $(1 - \lambda)^n \in R_{1-\lambda} = R_\lambda$  for all integers  $n > 0$ . If either  $|\lambda| < 1$  or  $|1 - \lambda| < 1$ , then  $R_\lambda$  is convex by Corollary 35, and thus  $\mathbb{R} \subseteq R_\lambda$  by Theorem 15. The only case left is when  $|\lambda| = |1 - \lambda| = 1$ . In this case,  $\lambda = (1 \pm i\sqrt{3})/2$ . Letting

$$\mu := \rho_{\lambda,1}(\lambda) = 2\lambda - \lambda^2 = \frac{3 \pm i\sqrt{3}}{2},$$

we have  $|\mu| > 1$ , and thus  $R_\mu$  is unbounded. But since  $\mu \in R_\lambda$ , we have  $R_\mu \subseteq R_\lambda$ , which makes  $R_\lambda$  unbounded.

If  $R_\lambda$  is unbounded, then so is  $Q_\lambda$ , because  $R_\lambda = \overline{Q_\lambda}$  by Lemma 7.  $\square$

**Lemma 40.**  *$R_\lambda$  is convex for all  $\lambda = x + iy$  where  $0 < x \leq 1/2$  and  $\sqrt{1 - x^2} < y \leq 1$ .*

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<sup>3</sup>i.e., a number of the form  $a + be^{i\tau/3}$  for some  $a, b \in \mathbb{Z}$



*Proof.* We know that  $\mu \in R_\lambda$ , where  $\mu := \rho_{1,\lambda}(\lambda) = 1 - \lambda + \lambda^2 = 1 - x + x^2 - y^2 + y(2x - 1)i$ . Letting  $\alpha := \Re(\mu) = 1 - x + x^2 - y^2$  and  $\beta := \Im(\mu) = y(2x - 1)$ , we have, for the values of  $x$  and  $y$  in question,

$$\begin{aligned} -x < x(x - 1) \leq \alpha < 2x^2 - x = x(2x - 1) \leq 0, \\ 2x - 1 \leq \beta \leq 0. \end{aligned}$$

Then  $0 < \alpha^2 + \beta^2 < (-x)^2 + (2x - 1)^2 = 5x^2 - 4x + 1 < 1$ , giving  $0 < |\mu| < 1$ . It follows from Corollary 36 that  $R_\lambda$  is convex.  $\square$

**Proposition 41.**  *$R_\lambda$  is convex for all  $\lambda = x + iy$  where  $0 < x < 1$  and  $-1 \leq y \leq 1$ , except for the two points  $e^{i\pi/6}$  and  $e^{-i\pi/6}$ .*

*Proof.* We just need to notice that the rectangular region given in the proposition is included in the union of a handful of other known subregions of  $\mathcal{C}$  (see Definition 17). Let  $\lambda$  be as in the proposition. If  $|\lambda| < 1$  or  $|1 - \lambda| < 1$ , then  $R_\lambda$  is convex by Corollary 35. If  $|\lambda| = 1$  or  $|1 - \lambda| = 1$ , then  $R_\lambda$  is convex by Proposition 37 and the fact that  $R_\lambda = R_{1-\lambda}$ . Let  $W$  be the wedge-shaped region of Lemma 40. Then the rest of the possible values of  $\lambda$  are covered by either  $W$ ,  $1 - W$ ,  $W^*$ , or  $1 - W^*$ , which all yield convex  $R_\lambda$  by Lemma 40 and Facts 9 and 10.  $\square$

## 5 Some $\lambda$ such that $R_\lambda = \mathbb{R}$

In this section we establish that  $R_\lambda$  is convex (and thus  $R_\lambda = \mathbb{R}$ ) for various  $\lambda \in \mathbb{R} - [0, 1]$ . We can assume without loss of generality that  $\lambda > 1$ , since  $R_\lambda = R_{1-\lambda}$ . If  $1 < \lambda < 2$ , then we already know that  $R_\lambda$  is convex by Corollary 35, and if  $\lambda \in \mathbb{Z}$ , then  $R_\lambda \subseteq \mathbb{Z}$  and thus is not convex. So we investigate the case where  $\lambda > 2$  and  $\lambda \notin \mathbb{Z}$ . At one point in time, we conjectured that  $R_\lambda$  is convex for all  $\lambda$  strictly between 2 and 3, but this turns out not to be the case, and the *unique* counterexample—where  $\lambda = 1 + \varphi \approx 2.618\dots$  where  $\varphi := (1 + \sqrt{5})/2$  is the Golden Ratio—gives a surprisingly strange discrete set  $R_\lambda$ .

We first quote a theorem that we prove later in Section 7. Recall the definition of  $\mathcal{C}$  in Definition 17.

**Theorem 42.**  *$\mathcal{C}$  is open.*

We defer the proof to Section 7, but we will use this theorem shortly.

**Lemma 43.** *If  $2 < \lambda < 1 + \sqrt{2}$ , then  $R_\lambda$  is convex.*

*Proof.* Consider  $\mu := \rho_{\lambda,1}(\lambda) = (1 - \lambda)\lambda + \lambda = 2\lambda - \lambda^2$ . We have  $\mu \in R_\lambda$ , and it is easy to check that if  $2 < \lambda < 1 + \sqrt{2}$ , then  $-1 < \mu < 0$ . Thus  $R_\mu$  is convex by Corollary 35, and so  $R_\lambda$  is convex.  $\square$

**Lemma 44.**  *$R_{1+\sqrt{2}}$  is convex.*

*Proof.* Setting  $\lambda := 1 + \sqrt{2}$ , we have that  $\rho_{\lambda,1}(\lambda) = 2\lambda - \lambda^2 = -1 \in R_\lambda$ . This implies that  $\rho_{-1,0}(\lambda) = \lambda - 1 = \sqrt{2} \in R_\lambda$ . Since  $1 < \sqrt{2} < 2$ , we have that  $R_{\sqrt{2}}$  is convex, which implies that  $R_\lambda$  is convex.  $\square$

**Corollary 45.** *There exists  $\varepsilon > 0$  such that  $R_\lambda$  is convex for all  $\lambda$  such that  $2 < \lambda < 1 + \sqrt{2} + \varepsilon$ .*

*Proof.* We have  $1 + \sqrt{2} \in \mathcal{C}$  by Lemma 44. Since  $\mathcal{C}$  is open (Theorem 42), there exists an open neighborhood of  $1 + \sqrt{2}$  contained in  $\mathcal{C}$ . The rest of the lemma follows from Lemma 43.  $\square$

Let  $\varphi := (1 + \sqrt{5})/2 \approx 1.618\dots$  be the Golden Ratio. We have  $\varphi^2 = 1 + \varphi$  and  $1/\varphi = \varphi - 1$ . The proof of the next proposition uses the fact that  $\mathcal{C}$  is open (Theorem 42).

**Proposition 46.** *If  $2 < \lambda < 1 + \varphi$ , then  $R_\lambda$  is convex.*

*Proof.* Suppose there exists some  $\mu \in ]2, 1 + \varphi[$  such that  $R_\mu$  is not convex. Then the set

$$X := \{\mu \mid 2 < \mu < 1 + \varphi \text{ \& } R_\mu \text{ is not convex}\}$$

is nonempty. Let  $\beta := \inf X$  be the greatest lower bound of  $X$ . We have  $\beta < 1 + \varphi$ , and Corollary 45 implies that  $\beta > 1 + \sqrt{2}$ . Now consider  $R_\beta$ . Setting  $\nu := 1 - \rho_{\beta,1}(\beta) = (\beta - 1)^2$ , we have  $\nu \in 1 - R_\beta = R_\beta$ . Furthermore, since  $1 + \sqrt{2} < \beta < 1 + \varphi$ , it is easy to check that  $2 < \nu < \beta$ . Since  $\beta = \inf X$ , this implies that  $R_\nu$  is convex, which in turn implies that  $R_\beta$  is convex, i.e.,  $\beta \in \mathcal{C}$ . But now since  $\mathcal{C}$  is open, there exists an open neighborhood surrounding  $\beta$  contained in  $\mathcal{C}$ , which contradicts the fact that  $\beta = \inf X$ . Thus  $X$  must be empty.  $\square$

**Proposition 47.** *If  $1 + \varphi < \lambda < 3$ , then  $R_\lambda$  is convex.*

*Proof.* For all  $\lambda \in [1 + \varphi, 3]$  define the real-valued functions

$$\begin{aligned} x_1(\lambda) &:= \rho_{\lambda,1}(\lambda) = (1 - \lambda)\lambda + \lambda = 2\lambda - \lambda^2, \\ x_2(\lambda) &:= 1 - x_1(\lambda) = 1 - 2\lambda + \lambda^2 = (1 - \lambda)^2, \\ \xi(\lambda) &:= \rho_{x_2(\lambda),\lambda}(\lambda) = (1 - \lambda)(1 - \lambda)^2 + \lambda^2 = (1 - \lambda)^3 + \lambda^2. \end{aligned}$$

The closure properties of  $R_\lambda$  imply  $\xi(\lambda) \in R_\lambda$  for all  $\lambda$ . It is routine to check that  $\xi$  is strictly monotone decreasing in the interval  $[1 + \varphi, 3]$ , and that  $\xi(1 + \varphi) = 1 + \varphi$  and  $\xi(3) = 1$ . Thus if  $1 + \varphi < \lambda < 3$ , we have  $1 < \xi(\lambda) < 1 + \varphi$ , and so by previous results,  $R_{\xi(\lambda)}$  is convex—implying that  $R_\lambda$  is convex by Corollary 25—except for the unique  $\lambda$  such that  $\xi(\lambda) = 2$ , which we consider separately.

Now assume that  $\lambda$  is the unique root of the equation  $\xi(\lambda) = 2$  such that  $1 + \varphi < \lambda < 3$ . Note that in this case,  $2 = \xi(\lambda) \in R_\lambda$ , and thus, owing again to the closure properties of  $R_\lambda$ , the point  $\mu := \rho_{\lambda,2}(2) = (1 - 2)\lambda + 2 \cdot 2 = 4 - \lambda$  is also in  $R_\lambda$ . Clearly,  $1 < \mu < 2$ , and thus  $R_\mu$  is convex, implying that  $R_\lambda$  is convex by Corollary 25.  $\square$

## 6 $R_{1+\varphi}$ is not convex

The next proposition shows that  $R_{1+\varphi}$  is not convex.

**Proposition 48.**

$$R_{1+\varphi} = \left\{ a + b\varphi : a, b \in \mathbb{Z} \text{ \& } \frac{b}{\varphi} \leq a \leq \frac{b}{\varphi} + 1 \right\} = \{1\} \cup \left\{ \left\lceil \frac{b}{\varphi} \right\rceil + b\varphi : b \in \mathbb{Z} \right\}.$$

*In particular,  $R_{1+\varphi}$  is discrete, and except for 0 and 1, any two adjacent points of  $R_{1+\varphi}$  differ either by  $\varphi$  or by  $1 + \varphi$ .*

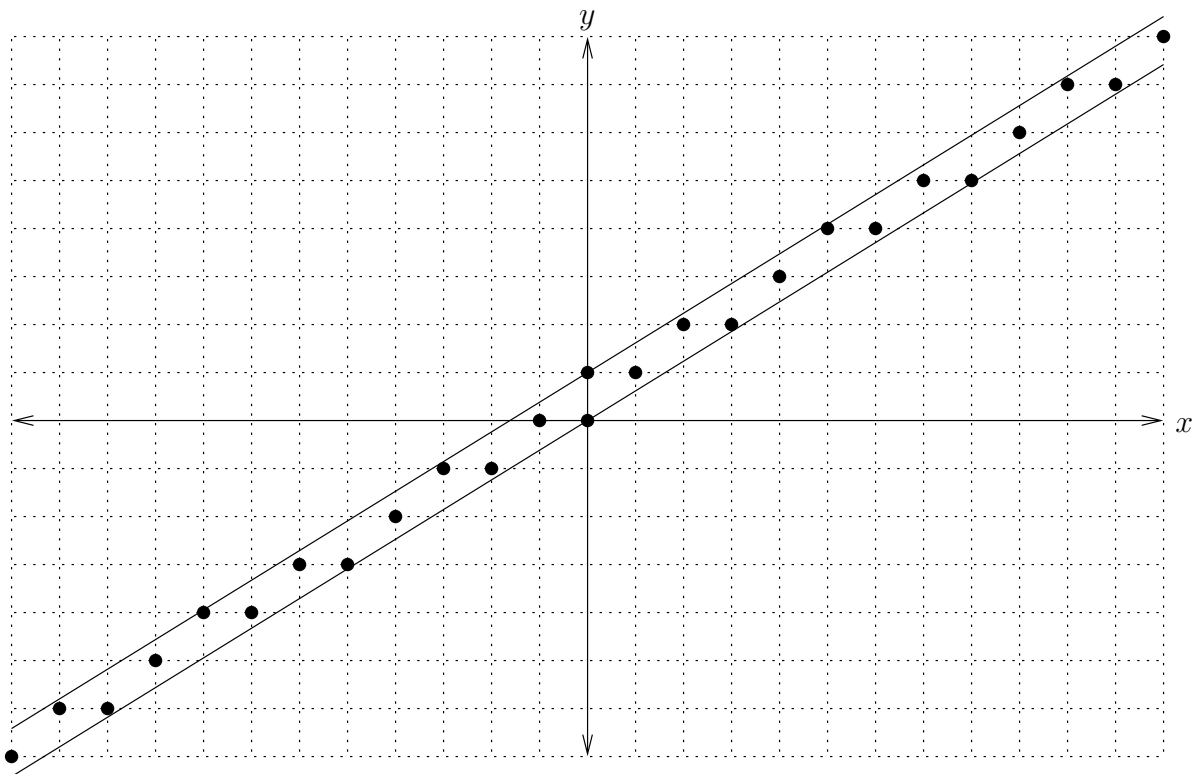


Figure 7: The points  $(b, a) \in \mathbb{Z} \times \mathbb{Z}$  such that  $a + b\varphi \in R_{1+\varphi}$  are shown. They are all the lattice points lying in the closed strip bounded by the lines  $y = x/\varphi$  and  $y = x/\varphi + 1$  (also shown). The figure illustrates the fact that  $R_{1+\varphi}$  contains no infinite arithmetic progressions, because any two points are connected either by the  $y$ -axis or by a line with rational slope, and this line eventually leaves the strip.

The set of pairs  $(b, a)$  such that  $a + b\varphi \in R_{1+\varphi}$  is illustrated in Figure 7. Although we give a complete, self-contained proof here, the first containment we show below—that  $R_{1+\varphi}$  is a subset of the right-hand side—is actually a special case of a more general result (Theorem 69) we prove in Section 10.

*Proof of Proposition 48.* The second equality is obvious, because  $\varphi$  is irrational. For the first equality, let

$$S := \{a + b\varphi : a, b \in \mathbb{Z} \text{ \& } b/\varphi \leq a \leq b/\varphi + 1\} .$$

We show that  $R_{1+\varphi} = S$  via two containments.

$R_{1+\varphi} \subseteq S$ : It suffices to show that  $S$  is  $(1 + \varphi)$ -convex, since  $\{0, 1\} \subseteq S$  and  $S$  is closed. Let  $x, y \in S$  be arbitrary, and write  $x = a + b\varphi$  and  $y = c + d\varphi$  for some  $a, b, c, d \in \mathbb{Z}$  such that  $b/\varphi \leq a \leq b/\varphi + 1$  and  $d/\varphi \leq c \leq d/\varphi + 1$ . Then we have, recalling that  $\varphi^2 = 1 + \varphi$ ,

$$\rho_{x,y}(1 + \varphi) = -\varphi x + (1 + \varphi)y = (c + d - b) + (c + 2d - a - b)\varphi = m + n\varphi ,$$

for integers  $m$  and  $n$  such that

$$m = c + d - b , \quad n = c + 2d - a - b .$$

Let  $0 \leq \delta \leq 1$  and  $0 \leq \varepsilon \leq 1$  be such that  $a = b/\varphi + \delta = b(\varphi - 1) + \delta$  and  $c = d/\varphi + \varepsilon = d(\varphi - 1) + \varepsilon$ . Then

$$\begin{aligned} m - \frac{n}{\varphi} &= m - n(\varphi - 1) = (c + d - b) - (c + 2d - a - b)(\varphi - 1) \\ &= a(\varphi - 1) + b(\varphi - 2) + c(2 - \varphi) + d(3 - 2\varphi) \\ &= (b(\varphi - 1) + \delta)(\varphi - 1) + b(\varphi - 2) + (d(\varphi - 1) + \varepsilon)(2 - \varphi) + d(3 - 2\varphi) \\ &= b((\varphi - 1)^2 + \varphi - 2) + d((\varphi - 1)(2 - \varphi) + 3 - 2\varphi) + \delta(\varphi - 1) + \varepsilon(2 - \varphi) \\ &= b(\varphi^2 - \varphi - 1) + d(1 + \varphi - \varphi^2) + \delta(\varphi - 1) + \varepsilon(2 - \varphi) = \delta(\varphi - 1) + \varepsilon(2 - \varphi) \\ &= \delta(\varphi - 1) + \varepsilon(1 - (\varphi - 1)) = \frac{\delta}{\varphi} + \frac{\varepsilon(\varphi - 1)}{\varphi} = \frac{\delta}{\varphi} + \frac{\varepsilon}{\varphi^2} = \frac{\varepsilon + \delta\varphi}{\varphi^2} = \frac{\varepsilon + \delta\varphi}{1 + \varphi} \in [0, 1] , \end{aligned}$$

and thus  $\rho_{x,y}(1 + \varphi) \in S$ . This shows that  $S$  is  $(1 + \varphi)$ -convex, and thus  $R_{1+\varphi} \subseteq S$ .

$S \subseteq R_{1+\varphi}$ : It is enough to show that  $\lceil b/\varphi \rceil + b\varphi \in R_{1+\varphi}$  for all  $b \in \mathbb{Z}$ . We show this by induction on  $|b|$ . For  $b \in \{-1, 0, 1\}$  this is easily checked; in particular,  $1 + \varphi = \rho_{0,1}(1 + \varphi)$  and  $-\varphi = \rho_{1,0}(1 + \varphi)$ . Thus we can start the induction with  $|b| \geq 2$ .

Notice that for all  $x \in \mathbb{Z} - \{0\}$ ,

$$\left\lceil \frac{-x}{\varphi} \right\rceil + (-x)\varphi = 1 - \left( \left\lceil \frac{x}{\varphi} \right\rceil + x\varphi \right) ,$$

which implies that the left-hand side is in  $R_{1+\varphi}$  if and only if the right-hand side is in  $R_{1+\varphi}$ , which is true if and only if  $\lceil x/\varphi \rceil + x\varphi \in R_{1+\varphi}$ . From this fact, we can assume WLOG that  $b \geq 2$ , the result for  $-b$  following immediately.

Assume  $b \in \mathbb{Z}$  and  $b \geq 2$ . Set  $a := \lfloor (b + 1)/\varphi \rfloor$ . We have  $1 \leq a < b$ , and so by the inductive hypothesis, both  $\lceil a/\varphi \rceil + a\varphi$  and  $\lceil -a/\varphi \rceil - a\varphi$  are in  $R_{1+\varphi}$ . Then letting  $y := \lceil -a/\varphi \rceil - a\varphi$ , the

following two values are both elements of  $R_{1+\varphi}$ :

$$\begin{aligned}\rho_{y,0}(1+\varphi) &= -\varphi y = -\varphi \left( \left\lceil \frac{-a}{\varphi} \right\rceil - a\varphi \right) = -\varphi(\lceil -a(\varphi-1) \rceil - a\varphi) = -\varphi(\lceil -a\varphi \rceil + a - a\varphi) \\ &= -\varphi(-\lfloor a\varphi \rfloor + a - a\varphi) = \varphi\lfloor a\varphi \rfloor - a\varphi + a\varphi^2 = \varphi\lfloor a\varphi \rfloor + a = a + (\lfloor a\varphi \rfloor - 1)\varphi, \\ \rho_{y,1}(1+\varphi) &= -\varphi y + 1 + \varphi = a + (\lfloor a\varphi \rfloor - 1)\varphi + 1 + \varphi = a + 1 + \lfloor a\varphi \rfloor \varphi.\end{aligned}$$

By the definition of  $a$ , we have  $b-1 < b+1-\varphi < a\varphi < b+1$ , and so the following two cases are exhaustive:

**Case 1:**  $\lceil a\varphi \rceil = b+1$ . Then  $\rho_{y,0}(1+\varphi) = a + b\varphi \in R_{1+\varphi}$ . Furthermore, in this case, we have  $b < a\varphi < b+1$ , and thus

$$\frac{b}{\varphi} < a < \frac{b+1}{\varphi} < \frac{b}{\varphi} + 1,$$

and so  $a = \lceil b/\varphi \rceil$  as desired.

**Case 2:**  $\lceil a\varphi \rceil = b$ . Then  $\rho_{y,1}(1+\varphi) = a + 1 + b\varphi \in R_{1+\varphi}$ . Furthermore, in this case, we have  $b-1 < a\varphi < b$ , and thus

$$\frac{b}{\varphi} - 1 < \frac{b-1}{\varphi} < a < \frac{b}{\varphi}.$$

Adding 1 to both sides gives

$$\frac{b}{\varphi} < a + 1 < \frac{b}{\varphi} + 1,$$

and so  $a + 1 = \lceil b/\varphi \rceil$  as desired.

The case for  $-b$  follows immediately as described above. This finishes the induction.  $\square$

$R_{1+\varphi}$  is a rather strange, irregular set. The following corollary implies that  $R_{1+\varphi}$  possesses no translational symmetry, and neither does any nonempty subset of  $R_{1+\varphi}$ .

**Corollary 49.**  *$R_{1+\varphi}$  contains no infinite arithmetic progressions.*

*Proof.* Suppose  $x, x+d, x+2d, x+3d, \dots \in R_{1+\varphi}$  for some  $x \in \mathbb{R}$  and  $d \in \mathbb{R} - \{0\}$ . Then  $d = m + n\varphi$  for some  $m, n \in \mathbb{Z}$ , and either  $x = 1$  or  $x = \lceil b/\varphi \rceil + b\varphi$  for some  $b \in \mathbb{Z}$ . We must have  $n \neq 0$ , for otherwise  $m \neq 0$ , and thus  $x + 2d$  is either  $1 + 2m$  or  $\lceil b/\varphi \rceil + 2m + b\varphi$ , neither of which is in  $R_{1+\varphi}$ . Then we have  $m \neq n/\varphi$  because  $n \neq 0$  and  $\varphi$  is irrational. This implies that by choosing  $k \in \mathbb{Z}^+$  large enough, we can make  $k|m - n/\varphi| = |km - kn/\varphi|$  as large as desired. Then, for sufficiently large  $k$  depending on  $x$ , if  $x = 1$ , we have

$$1 \neq x + kd = (1 + km) + kn\varphi \neq \left\lceil \frac{kn}{\varphi} \right\rceil + kn\varphi,$$

and if  $x = \lceil b/\varphi \rceil + b\varphi$  for some  $b \in \mathbb{Z}$ , then

$$1 \neq x + kd = \left\lceil \frac{b}{\varphi} \right\rceil + km + (b + kn)\varphi \neq \left\lceil \frac{b + kn}{\varphi} \right\rceil + (b + kn)\varphi,$$

and so in either case,  $x + kd \notin R_{1+\varphi}$ . Contradiction.  $\square$

## 7 $\mathcal{C}$ is open

In this section, we show, among other things, that  $\mathcal{C}$  is open, but beforehand, we introduce some new facts and concepts that will also be useful elsewhere. Recall the definition of  $Q_\lambda$  in Definition 2.

**Definition 50.** Let  $x$  be a formal indeterminate, and let  $Q_{[x]}$  denote the least set of polynomials such that

1. The constant polynomials 0 and 1 are both in  $Q_{[x]}$ , and
2. For every  $S, T \in Q_{[x]}$ ,  $(1-x)S + xT \in Q_{[x]}$ .

Note that  $Q_{[x]} \subseteq \mathbb{Z}[x]$ . In Section 11, we will obtain some further constraints on the elements of  $Q_{[x]}$ , including upper bounds on the number of elements of  $Q_{[x]}$  of degree  $\leq n$ , for  $n = 0, 1, 2, \dots$ .

**Definition 51.** A *derivation* of a polynomial  $P \in \mathbb{Z}[x]$  is a nonempty, finite sequence  $P_1, P_2, \dots, P_n$  of polynomials in  $\mathbb{Z}[x]$  such that  $P = P_n$  and for all  $1 \leq k \leq n$ ,  $P_k$  is either 0 or 1 or  $(1-x)P_i + xP_j$  for some  $1 \leq i, j < k$ . Such a derivation has *length*  $n$ .

**Fact 52.** A polynomial  $P \in \mathbb{Z}[x]$  is in  $Q_{[x]}$  if and only if  $P$  has a derivation.

**Lemma 53.** For any  $\lambda \in \mathbb{C}$ ,  $Q_\lambda = \{P(\lambda) \mid P \in Q_{[x]}\}$ .

*Proof.* The  $\subseteq$ -containment clearly holds because  $\{P(\lambda) \mid P \in Q_{[x]}\}$  is  $\lambda$ -convex. The  $\supseteq$ -containment, i.e., that  $P(\lambda) \in Q_\lambda$  for all  $P \in Q_{[x]}$ , is proved by a routine induction on the length of a derivation of  $P$ : If either  $P = 0$  or  $P = 1$ , then  $P(\lambda) \in \{0, 1\} \subseteq Q_\lambda$ , and we are done. Otherwise,  $P = (1-x)S + xT$ , where  $S$  and  $T$  have derivations shorter than that of  $P$ . By the inductive hypothesis,  $S(\lambda) \in Q_\lambda$  and  $T(\lambda) \in Q_\lambda$ , and so  $P(\lambda) = (1-\lambda)S(\lambda) + \lambda T(\lambda) = \rho_{S(\lambda), T(\lambda)}(\lambda) \in Q_\lambda$ .  $\square$

**Corollary 54.**  $Q_\lambda$  is countable for all  $\lambda \in \mathbb{C}$ .

Now we can prove Theorem 42.

*Proof of Theorem 42.* Let  $D := \{x \in \mathbb{C} : 0 < |x| < 1 \vee 0 < |1-x| < 1\}$ . Note that  $D$  is open. For any  $\lambda \in \mathbb{C}$ , we have

$$\begin{aligned}
 R_\lambda \text{ is convex} &\iff R_\lambda \cap D \neq \emptyset && \text{(Corollary 36)} \\
 &\iff Q_\lambda \cap D \neq \emptyset && \text{(Lemma 7)} \\
 &\iff (\exists P \in Q_{[x]})(P(\lambda) \in D) && \text{(Lemma 53)} \\
 &\iff \lambda \in \bigcup_{P \in Q_{[x]}} P^{-1}(D).
 \end{aligned}$$

Thus  $\mathcal{C} = \bigcup_{P \in Q_{[x]}} P^{-1}(D)$ , which is the union of open sets, because each  $P \in Q_{[x]}$  is a continuous map  $\mathbb{C} \rightarrow \mathbb{C}$ . Thus  $\mathcal{C}$  is open.  $\square$

We end this section with the easy result that  $\mathcal{C}$  “surrounds” all roots of polynomials in  $Q_{[x]}$ .

**Proposition 55.** For any  $P \in Q_{[x]}$  and any complex root  $r$  of  $P$ , there exists an open neighborhood  $N$  of  $r$  such that  $N - \{r\} \subseteq \mathcal{C}$ .

*Proof.* By the continuity of  $P$ , there exists a neighborhood  $N'$  of  $r$  such that  $|P(x)| < 1$  for all  $x \in N'$ . Since  $P$  has only finitely many roots, there exists an  $\varepsilon > 0$  such that  $0 < P(x) < 1$  for all  $x$  such that  $0 < |x - r| < \varepsilon$ . Letting  $N := \{x \in \mathbb{C} : |x - r| < \varepsilon\}$ , we have  $0 < P(x) < 1$  for all  $x \in N - \{r\}$ . For these  $x$ , we know that  $R_{P(x)}$  is convex by Corollary 35. Since  $P(x) \in R_x$ , we then have that  $R_x$  is convex by Corollary 25. Thus  $x \in \mathcal{C}$  for all  $x \in N - \{r\}$ .  $\square$

## 8 $R_\lambda$ when $\Re(\lambda) = 1/2$

Here we look at  $R_\lambda$  for some  $\lambda$  with real part  $1/2$ . For these  $\lambda$ , we have  $1 - \lambda = \lambda^*$ , the complex conjugate of  $\lambda$ , and so  $R_\lambda$  is closed under complex conjugate. Furthermore,  $R_\lambda$  is convex iff  $R_{\lambda^*}$  is convex. We also have in particular,

$$\rho_{\lambda,0}(\lambda) = \lambda(1 - \lambda) = |\lambda|^2,$$

and so  $|\lambda|^2 \in R_\lambda$ .

**Proposition 56.** *Suppose  $\Re(\lambda) = 1/2$  and  $|\lambda| < \varphi$ . Then  $R_\lambda$  is convex if and only if  $|\lambda| \notin \{1, \sqrt{2}\}$ .*

*Proof.* First, suppose  $|\lambda| \notin \{1, \sqrt{2}\}$ . We have  $|\lambda|^2 \in [1/4, 1 + \varphi[ - \{1, 2\}$ , and thus  $R_{|\lambda|^2}$  is convex by previous results. Since  $|\lambda|^2 \in R_\lambda$ , we have that  $R_\lambda$  is convex for these  $\lambda$ .

If  $|\lambda| = 1$ , then  $\lambda = (1 \pm i\sqrt{3})/2$ , which is an Eisenstein integer, and thus  $R_\lambda$  consists only of Eisenstein integers, and so is discrete.

If  $|\lambda| = \sqrt{2}$ , then  $\lambda = (1 \pm i\sqrt{7})/2$ . In this case,  $\lambda$  is a nonreal algebraic integer of degree 2. In fact,  $\lambda$  is a root of the monic, quadratic polynomial  $x^2 - x + 2 \in \mathbb{Z}[x]$ , which is irreducible over  $\mathbb{Q}$ . Thus  $R_\lambda \subseteq \mathbb{Z}[\lambda] = \{a + b\lambda \mid a, b \in \mathbb{Z}\}$ , which is a discrete subring of  $\mathbb{C}$ .  $\square$

**Corollary 57.** *Suppose  $\Re(\lambda) = 1/2$ , and let  $y := |\Im(\lambda)|$ . If  $y < \sqrt{5 + 2\sqrt{5}}/2$  and  $y \notin \{\sqrt{3}/2, \sqrt{7}/2\}$ , then  $R_\lambda$  is convex.*

What if  $\lambda = (1 + i\sqrt{5 + 2\sqrt{5}})/2$ ? We have  $|\lambda| = \varphi$  in this case, and in fact,  $\lambda = \varphi e^{i\tau/5}$ . The points  $0, 1, \lambda$  form the vertices of an acute Robinson triangle, i.e., a triangle with sides  $(1, \varphi, \varphi)$ . The next proposition may come as a surprise, given what we know about  $R_{|\lambda|^2} = R_{1+\varphi}$ .

**Proposition 58.**  *$R_\lambda$  is convex, where  $\lambda = (1 + i\sqrt{5 + 2\sqrt{5}})/2 = \varphi e^{i\tau/5}$ .*

*Proof.* We show via an explicit derivation that the point  $\mu := e^{i\tau(7/10)}$  is in  $R_\lambda$ . (The derivation below was found by a computer-assisted search.) It then follows from Corollary 25 and Proposition 37 that  $R_\lambda$  is convex.

We first note that  $\lambda$  is an algebraic integer of degree 4 with minimum polynomial  $x^4 - 2x^3 + 4x^2 - 3x + 1$ . Thus

$$\lambda^4 = -1 + \lambda - 4\lambda^2 + 2\lambda^3. \quad (2)$$

It can also be readily checked (on purely geometric grounds, even) that  $\mu = 2\lambda - \lambda^2 + \lambda^3$ . The

derivation of  $\mu$  follows:

$$\begin{aligned}
x_0 &:= 0 \\
x_1 &:= 1 \\
x_2 &:= \rho_{x_0, x_1}(\lambda) = \lambda \\
x_3 &:= \rho_{x_1, x_0}(\lambda) = 1 - \lambda \\
x_4 &:= \rho_{x_0, x_2}(\lambda) = \lambda^2 \\
x_5 &:= \rho_{x_2, x_1}(\lambda) = \lambda(2 - \lambda) = 2\lambda - \lambda^2 \\
x_6 &:= \rho_{x_1, x_3}(\lambda) = (1 + \lambda)(1 - \lambda) = 1 - \lambda^2 \\
x_7 &:= \rho_{x_4, x_0}(\lambda) = \lambda^2(1 - \lambda) = \lambda^2 - \lambda^3 \\
x_8 &:= \rho_{x_1, x_6}(\lambda) = (1 + \lambda + \lambda^2)(1 - \lambda) = 1 - \lambda^3 \\
x_9 &:= \rho_{x_7, x_5}(\lambda) = \lambda^2(3 - 3\lambda + \lambda^2) = -1 + 3\lambda - \lambda^2 - \lambda^3 & \text{(using (2))} \\
x_{10} &:= \rho_{x_2, x_8}(\lambda) = \lambda(1 - \lambda)(2 + \lambda + \lambda^2) = 1 - \lambda + 3\lambda^2 - 2\lambda^3 & \text{(using (2))} \\
\mu = x_{11} &:= \rho_{x_9, x_{10}}(\lambda) = \lambda^2(1 - \lambda)(5 - 2\lambda + 2\lambda^2) = 2\lambda - \lambda^2 + \lambda^3 & \text{(using (2))}
\end{aligned}$$

□

If  $\lambda = (1 - i\sqrt{5 + 2\sqrt{5}})/2$ , then  $R_\lambda$  is also clearly convex by symmetry. We can now strengthen Proposition 56.

**Corollary 59.** *Suppose  $\Re(\lambda) = 1/2$  and  $|\lambda| \leq \sqrt{3}$ . Then  $R_\lambda$  is convex if and only if  $|\lambda| \notin \{1, \sqrt{2}, \sqrt{3}\}$ .*

*Proof.* All that remains is to observe that if  $|\lambda| = \sqrt{3}$ , then  $\lambda = (1 \pm i\sqrt{11})/2$  is a nonreal algebraic number of degree 2 with minimal polynomial  $x^2 - x + 3$ , and thus  $\mathbb{Z}[\lambda]$  is a discrete subring of  $\mathbb{C}$  containing  $R_\lambda$ . □

## 9 $R_\lambda$ for $\lambda$ contained in a subring of $\mathbb{C}$

In this section, consider the case where  $\lambda$  belongs to a discrete subring of  $D$ . We start with an easy lemma that will also be used in Section 10.

**Lemma 60.** *Suppose  $D$  is a subring of  $\mathbb{C}$  that is discrete in the induced topology. Then  $D$  cannot contain two distinct points less than unit distance apart. Consequently,  $D$  is (topologically) closed, and  $D \cap \mathbb{R} = \mathbb{Z}$ .*

*Proof.* Suppose for the sake of contradiction that  $a, b \in D$  are such that  $0 < |a - b| < 1$ . Then  $(a - b)^n \in D - \{0\}$  for all integers  $n > 0$ , and  $\lim_{n \rightarrow \infty} (a - b)^n = 0$ . This means that  $0 \in D$  is an accumulation point of  $D$ , and hence  $D$  is not discrete. The other two consequences follow immediately. □

From Fact 38 and Lemma 60 it follows that if  $D$  is discrete and  $\lambda \in D$ , then  $R_\lambda = Q_\lambda \subseteq D$  and is discrete as well. Sometimes equality holds in the inclusion above. For example,

**Fact 61.**  $R_2 = R_{-1} = \mathbb{Z}$ .



Usually, equality does not hold;  $R_2$  is the only case where equality holds for  $D := \mathbb{Z}$ .  $R_\lambda$  is a proper subset of  $\mathbb{Z}$  for all  $\lambda \geq 3$ , as the next general lemma implies.

**Lemma 62.** *Let  $D$  be any subring of  $\mathbb{C}$ . For any  $\lambda \in D$ , let  $I_\lambda := \{a\lambda(1-\lambda) \mid a \in D\}$  be the ideal of  $D$  generated by  $\lambda(1-\lambda)$ . Then*

$$Q_\lambda \subseteq I_\lambda \cup (I_\lambda + 1) \cup (I_\lambda + \lambda) \cup (I_\lambda + 1 - \lambda) .$$

*If  $D$  is discrete, then the same inclusion holds for  $R_\lambda$ .*

*Proof sketch.* One merely checks that the right-hand side is  $\lambda$ -convex. □

**Corollary 63.** *Let  $D$  and  $\lambda$  be as in Lemma 62, above. Then*

$$Q_\lambda \subseteq \{a\lambda + b \mid a \in D \ \& \ b \in \{0, 1\}\} \cap \{c(1-\lambda) + d \mid c \in D \ \& \ d \in \{0, 1\}\} .$$

*If  $D$  is discrete, then the same inclusion holds for  $R_\lambda$ .*

When applying Lemma 62 with  $D := \mathbb{Z}$ , it suffices to consider  $\lambda > 1$ , so in this case, we will assume  $\lambda \geq 2$ .

**Corollary 64.** *For all  $\lambda \in \mathbb{Z}$  such that  $\lambda \geq 2$  and all  $n \in \mathbb{Z}$ , if  $n \in R_\lambda$ , then either*

$$\begin{aligned} n &\equiv 0 \pmod{\lambda(\lambda-1)} \text{ or} \\ n &\equiv 1 \pmod{\lambda(\lambda-1)} \text{ or} \\ n &\equiv \lambda \pmod{\lambda(\lambda-1)} \text{ or} \\ n &\equiv 1-\lambda \pmod{\lambda(\lambda-1)} . \end{aligned}$$

*Equivalently, if  $n$  is in  $R_\lambda$  then  $n$  is congruent to either 0 or 1 modulo both  $\lambda$  and  $\lambda-1$ . In particular, if  $n \in R_\lambda$ , then  $n \equiv n^2 \pmod{\lambda(\lambda-1)}$ .*

One would generally like to know when equality holds in Lemma 60 for discrete  $D$ . We conjecture that it holds at least for  $D := \mathbb{Z}$  (Conjecture 99 in Section 14).

We can at least prove a sufficient condition for equality (Theorem 67, below). First, we will say that a point  $x \in \mathbb{C}$  is a *translation point* of  $R_\lambda$  iff  $\{x, x+1\} \subseteq R_\lambda$ .

**Lemma 65.** *If  $x$  is a translation point for  $R_\lambda$ , then so is  $-x$ , and furthermore,  $\rho_{x,x+1}(R_\lambda) = \rho_{-x,-x+1}(R_\lambda) = R_\lambda$ .*

*Proof.*  $R_\lambda = 1 - R_\lambda$  by Corollary 24, so if  $x$  is a translation point of  $R_\lambda$ , then so is  $-x$ . We have  $\rho_{x,x+1}(R_\lambda) \subseteq R_\lambda$  and  $\rho_{-x,-x+1}(R_\lambda) \subseteq R_\lambda$  by Corollary 20. To get the reverse containments, we observe that  $\rho_{x,x+1}$  and  $\rho_{-x,-x+1}$  are inverses of each other, and so, applying  $\rho_{-x,-x+1}$  to both sides of the first containment, we get

$$R_\lambda = \rho_{-x,-x+1}(\rho_{x,x+1}(R_\lambda)) \subseteq \rho_{-x,-x+1}(R_\lambda) ,$$

and applying  $\rho_{x,x+1}$  to the second containment similarly yields  $R_\lambda \subseteq \rho_{x,x+1}(R_\lambda)$ . □

**Corollary 66.** *For any  $\lambda \in \mathbb{C}$ , the translation points of  $R_\lambda$  form a  $\mathbb{Z}$ -submodule of  $\mathbb{C}$ . (That is, every integer linear combination of translation points is a translation point.)*

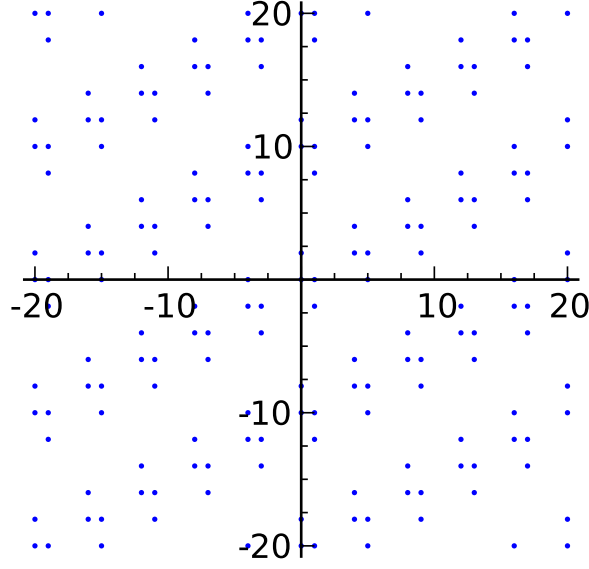


Figure 8: A plot of  $R_{2i}$ .

**Theorem 67.** *Let  $D$  be a discrete subring of  $\mathbb{C}$ , and let  $S \subseteq D$  span  $D$  as a  $\mathbb{Z}$ -module (that is, every element of  $D$  is a  $\mathbb{Z}$ -linear combination of elements from  $S$ ). Suppose  $\lambda \in D$  is such that  $a\lambda(1 - \lambda)$  is a translation point of  $R_\lambda$  for every  $a \in S$ . Then*

$$R_\lambda = I_\lambda \cup (I_\lambda + 1) \cup (I_\lambda + \lambda) \cup (I_\lambda + 1 - \lambda), \quad (3)$$

where  $I_\lambda \subseteq D$  is the ideal generated by  $\lambda(1 - \lambda)$ .

*Proof.* Since  $D$  is discrete, we have  $R_\lambda = Q_\lambda$ , and so the  $\subseteq$ -containment holds by Lemma 62. For the reverse containment, note that by the previous corollary, every element of  $I_\lambda$  is a translation point of  $R_\lambda$ . Now suppose  $x \in I_\lambda + b$  for some  $b \in \{0, 1, \lambda, 1 - \lambda\}$ . Note that  $b \in R_\lambda$ . Writing  $x := a + b$  for  $a \in I_\lambda$ , we have

$$x = \rho_{a,a+1}(b) \in \rho_{a,a+1}(R_\lambda) = R_\lambda$$

by Lemma 65, because  $a$  is a translation point of  $R_\lambda$ . □

Theorem 67 is useful because the rings in question are finitely generated  $\mathbb{Z}$ -modules, and so Equation (3) can be verified by testing a finite number of points. For example, Figure 8 shows  $R_{2i}$ . Equation (3) holds for  $\lambda := 2i$ , because  $\mathbb{Z}[2i]$  is spanned by  $\{1, 2i\}$ , and it is evident from the picture that both  $4 + 2i = \lambda(1 - \lambda)$  and  $-4 + 8i = 2i\lambda(1 - \lambda)$  are both translation points of  $R_{2i}$ .

**Corollary 68.** *For any  $\lambda \in \mathbb{Z}$  such that  $\lambda \geq 2$ , if  $\lambda(\lambda - 1) \in R_\lambda$ , then Equation (18) of Conjecture 99 holds.*

*Proof.* Along with 0 and 1, the following are all elements of  $R_\lambda$ , regardless of the hypothesis:

$$\begin{aligned}\rho_{0,1}(\lambda) &= \lambda, \\ \rho_{1,0}(\lambda) &= 1 - \lambda, \\ \rho_{\lambda,0}(\lambda) &= \lambda(1 - \lambda), \\ 1 - \lambda(1 - \lambda) &= \lambda(\lambda - 1) + 1.\end{aligned}$$

Thus  $\lambda(\lambda - 1)$  is a translation point of  $R_\lambda$ . Since  $\mathbb{Z}$  is spanned by  $\{1\}$ , the corollary follows by Theorem 67.  $\square$

We have used Corollary 68 to verify Conjecture 99 by hand for  $2 \leq \lambda \leq 6$  and by computer for  $7 \leq \lambda \leq 10$ .

## 10 $R_\lambda$ for some algebraic integers $\lambda$

In this section, we prove the following theorem, which gives some sufficient conditions for  $R_\lambda$  to be discrete for certain algebraic integers  $\lambda$ , including some (e.g.,  $1 + \varphi$ ) not belonging to any discrete subring of  $\mathbb{C}$ . In fact, all cases we currently know of where  $R_\lambda$  is discrete follow from this theorem.

**Theorem 69.** *Let  $D \subseteq \mathbb{C}$  be a discrete subring of  $\mathbb{C}$ , and let  $\lambda \in \mathbb{C}$  be a root of some monic polynomial  $p \in D[x]$  of degree  $d > 0$  such that  $p$  has no multiple roots in  $\mathbb{C}$ . Let  $\mu_0, \dots, \mu_{d-2} \in \mathbb{C}$  be the other roots of  $p$  besides  $\lambda$ . Then  $R_\lambda$  is discrete (in particular, not convex) when one of the following is true:*

1.  $\mu_0, \dots, \mu_{d-2}$  are all elements of  $[0, 1]$ .
2.  $\mu_0, \dots, \mu_{d-3}$  are all elements of  $[0, 1]$ ,  $\lambda$  is non-real, and  $D = \mathbb{Z}$ .

Let  $D$  be a discrete subring of  $\mathbb{C}$ .  $D$  is closed by Lemma 60. If  $\lambda$  belongs to  $D$ , then  $R_\lambda$  is discrete for an “easy” reason:  $R_\lambda \subseteq D$ . Proposition 48 gave us our first example of a discrete  $R_\lambda$  that is *not* contained in any discrete subring of  $\mathbb{C}$ . Theorem 69 will give us many other examples, i.e., values of  $\lambda$  such that  $R_\lambda$  is discrete but  $\mathbb{Z}[\lambda]$  is not.

To prove Theorem 69, we build up some more machinery using some results of linear algebra—especially the spectral decomposition of an operator—and this machinery may have more general application.

Throughout this section, for convenience, we start the indexing of vectors and matrices with 0 instead of with 1. If  $E$  is some expression of matrix type, we let  $[E]_{ij}$  denote the  $(i, j)$ th entry of  $E$ , for any appropriate integers  $i, j \geq 0$ .

We first extend our definition of  $\rho$  to vector spaces.

**Definition 70.** Let  $V$  be any vector space over some field  $k$ .

1. For any vectors  $u, v \in V$  and  $k$ -linear map  $\Lambda : V \rightarrow V$ , define

$$\rho_{u,v}(\Lambda) := (I - \Lambda)u + \Lambda v = u + \Lambda(v - u).$$

2. A set  $S \subseteq V$  is  $\Lambda$ -convex iff  $\rho_{u,v}(\Lambda) \in S$  whenever  $u, v \in S$ .

3. For any set of vectors  $S \subseteq V$ , we define  $Q_\Lambda(S) \subseteq V$  as the smallest  $\Lambda$ -convex superset of  $S$ .

We could easily define  $\Lambda$ -convexity and  $R_\Lambda(S)$  for appropriate vector spaces, but we won't need this notion here.

**Fact 71.** *Let  $V$ ,  $k$ , and  $\Lambda$  be as in Definition 70. For any  $u, v, w, x \in V$  and any  $a \in k$ , we have*

$$\rho_{u+v, w+x}(\Lambda) = \rho_{u, w}(\Lambda) + \rho_{v, x}(\Lambda)$$

and

$$\rho_{au, av}(\Lambda) = a\rho_{u, v}(\Lambda) .$$

**Fact 72.** *Let  $U$  and  $V$  be vector spaces over some field  $k$ , and let  $\Lambda_U : U \rightarrow U$ ,  $\Lambda_V : V \rightarrow V$ , and  $t : U \rightarrow V$  be  $k$ -linear maps such that  $\Lambda_V \circ t = t \circ \Lambda_U$ . Then for any  $u, v \in U$ ,*

$$t(\rho_{u, v}(\Lambda_U)) = \rho_{t(u), t(v)}(\Lambda_V) .$$

**Lemma 73.** *Let  $U$ ,  $V$ ,  $k$ ,  $\Lambda_U$ ,  $\Lambda_V$ , and  $t$  be as in Fact 72. For any  $S \subseteq U$ , we have*

$$t(Q_{\Lambda_U}(S)) = Q_{\Lambda_V}(t(S)) .$$

*Proof.* This is a bit similar to the proof of Lemma 8.

First we show  $\supseteq$ . We have  $S \subseteq Q_{\Lambda_U}(S)$  by definition, so applying  $t$  to both sides yields  $t(S) \subseteq t(Q_{\Lambda_U}(S))$ . Therefore, by the minimality of  $Q_{\Lambda_V}(t(S))$ , it is enough to show that  $t(Q_{\Lambda_U}(S))$  is  $\Lambda_V$ -convex. Given  $x, y \in t(Q_{\Lambda_U}(S))$ , pick  $u, v \in Q_{\Lambda_U}(S)$  such that  $x = t(u)$  and  $y = t(v)$ . Then  $\rho_{u, v}(\Lambda_U) \in Q_{\Lambda_U}(S)$ , and by Fact 72,  $\rho_{x, y}(\Lambda_V) = t(\rho_{u, v}(\Lambda_U)) \in t(Q_{\Lambda_U}(S))$ , and we are done.

Now we show  $\subseteq$ . For this it suffices that  $Q_{\Lambda_U}(S) \subseteq t^{-1}(Q_{\Lambda_V}(t(S)))$ . We have  $t(S) \subseteq Q_{\Lambda_V}(t(S))$  by definition, so applying  $t^{-1}$  to both sides, we get  $S \subseteq t^{-1}(t(S)) \subseteq t^{-1}(Q_{\Lambda_V}(t(S)))$ . So, as above, it remains to show that  $t^{-1}(Q_{\Lambda_V}(t(S)))$  is  $\Lambda_U$ -convex. Given  $u, v \in t^{-1}(Q_{\Lambda_V}(t(S)))$ , let  $x := t(u)$  and  $y := t(v)$ ; we have  $x, y \in Q_{\Lambda_V}(t(S))$ , and so  $\rho_{x, y}(\Lambda_V) \in Q_{\Lambda_V}(t(S))$  because  $Q_{\Lambda_V}(t(S))$  is  $\Lambda_V$ -convex. Using Fact 72 again, we have  $\rho_{x, y}(\Lambda_V) = t(\rho_{u, v}(\Lambda_U))$ , and thus  $\rho_{u, v}(\Lambda_U)$  is an element of  $t^{-1}(Q_{\Lambda_V}(t(S)))$ .  $\square$

Consider the case where  $V = \mathbb{C}^d$  for some finite  $d$  and where  $\Lambda : V \rightarrow V$  is a  $\mathbb{C}$ -linear map given by some *diagonal* matrix  $L$  with respect to the standard basis  $\{e_0, \dots, e_{d-1}\}$  of  $\mathbb{C}^d$ . For  $0 \leq i < d$ , let  $\mu_i := [L]_{ii}$  be the diagonal entries of  $L$ . So on the  $i$ th coordinate,  $\Lambda$  acts as scalar multiplication by  $\mu_i$ . The next lemma says that to get the  $\Lambda$ -convex closure of two vectors  $u$  and  $v$ , we can act coordinatewise, obtaining a translated copy of  $Q_{\mu_i}$  on the  $i$ th coordinate, and we can do this step by step, in analogy with Lemma 53.

Recall the definition of  $Q_{[x]}$  in Definition 50.

**Lemma 74.** *Let  $V$ ,  $\Lambda := L$ , and  $\mu_0, \dots, \mu_{d-1}$ , be as in the previous paragraph. For any two vectors  $u, v \in V$  with  $u = (u_0, \dots, u_{d-1})^\top$  and  $v = (v_0, \dots, v_{d-1})^\top$ ,*

$$Q_\Lambda(\{u, v\}) = \left\{ \sum_{i=0}^{d-1} \rho_{u_i, v_i}(p(\mu_i))e_i \mid p \in Q_{[x]} \right\} = \left\{ \sum_{i=0}^{d-1} (u_i + p(\mu_i)(v_i - u_i))e_i \mid p \in Q_{[x]} \right\} . \quad (4)$$

*Proof.* The second equality of (4) holds by definition. For the first equality, we first show  $\subseteq$  by verifying that the right-hand side contains  $u$  and  $v$  and is  $\Lambda$ -convex. Clearly,  $u = \sum_i u_i e_i = \sum_i \rho_{u_i, v_i}(0) e_i$  and  $v = \sum_i v_i e_i = \sum_i \rho_{u_i, v_i}(1) e_i$  are elements of the right-hand side, because the constant polynomials 0 and 1 are in  $Q[x]$ . To show  $\Lambda$ -convexity, let  $p, q \in Q[x]$  be arbitrary, and set  $a := \sum_i a_i e_i$  and  $b := \sum_i b_i e_i$ , where  $a_i := \rho_{u_i, v_i}(p(\mu_i))$  and  $b_i := \rho_{u_i, v_i}(q(\mu_i))$  for  $0 \leq i < d$ . Then setting  $r(x) := (1-x)p(x) + xq(x) \in Q[x]$ , we have, by Fact 71,

$$\begin{aligned} \rho_{a,b}(\Lambda) &= \sum_{i=0}^{d-1} \rho_{a_i e_i, b_i e_i}(\Lambda) = \sum_i (a_i e_i + \Lambda(b_i e_i - a_i e_i)) = \sum_i (a_i e_i + (b_i - a_i) \Lambda e_i) \\ &= \sum_i (a_i + \mu_i(b_i - a_i)) e_i = \sum_i \rho_{a_i, b_i}(\mu_i) e_i = \sum_i (\rho_{u_i, v_i} \circ \rho_{p(\mu_i), q(\mu_i)}) (\mu_i) e_i \\ &= \sum_i \rho_{u_i, v_i}((1 - \mu_i)p(\mu_i) + \mu_i q(\mu_i)) e_i = \sum_i \rho_{u_i, v_i}(r(\mu_i)) e_i. \end{aligned}$$

This shows that  $\rho_{a,b}(\Lambda)$  belongs to the right-hand side of (4), and thus the right-hand side is  $\Lambda$ -convex.

To show  $\supseteq$  in (4), we use a routine induction on the length of derivations. First, if  $p$  is the zero polynomial, we have  $\sum_{i=0}^{d-1} \rho_{u_i, v_i}(p(\mu_i)) e_i = \sum_i u_i e_i = u \in Q_\Lambda(\{u, v\})$ . Similarly, if  $p = 1$ , then we get  $\sum_{i=0}^{d-1} \rho_{u_i, v_i}(p(\mu_i)) e_i = v \in Q_\Lambda(\{u, v\})$ . Now fix  $n > 0$ , and assume that  $\sum_{i=0}^{d-1} \rho_{u_i, v_i}(p(\mu_i)) e_i \in Q_\Lambda(\{u, v\})$  for every  $p \in Q[x]$  with a derivation of length  $\leq n$ . Let  $r \in Q[x]$  be any polynomial with a derivation of length  $n+1$ . We write  $r(x) = (1-x)p(x) + xq(x)$  for some  $p, q \in Q[x]$ , each with derivations of length at most  $n$ . Then setting  $a_i := \rho_{u_i, v_i}(p(\mu_i))$  and  $b_i := \rho_{u_i, v_i}(q(\mu_i))$  for each  $i$ , we just run the above string of equations backwards to get  $\sum_{i=0}^{d-1} \rho_{u_i, v_i}(r(\mu_i)) e_i = \rho_{a,b}(\Lambda)$ , where  $a := \sum_i a_i e_i$  and  $b := \sum_i b_i e_i$  are elements of  $Q_\Lambda(\{u, v\})$  by the inductive hypothesis. Since  $Q_\Lambda(\{u, v\})$  is  $\Lambda$ -convex, we get  $\sum_{i=0}^{d-1} \rho_{u_i, v_i}(r(\mu_i)) e_i = \rho_{a,b}(\Lambda) \in Q_\Lambda(\{u, v\})$ , so the inclusion holds for  $r$ .  $\square$

We would like to apply Lemma 74 to the operators  $\Lambda_p$  of Definition 75, below, and these are clearly not diagonal. They are *diagonalizable*, however, that is, they have eigenbases (at least provided  $p$  has no multiple roots). We can apply Lemma 74 to some  $\Lambda_p$  after we transform to an eigenbasis with the help of Lemma 73.

**Definition 75.** Let  $p(x) = x^d + \sum_{j=0}^{d-1} c_j x^j$  be some monic polynomial in  $\mathbb{C}[x]$  of degree  $d > 0$  with coefficients  $c_0, \dots, c_{d-1} \in \mathbb{C}$ . Define the  $d \times d$  matrix

$$\Lambda_p := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ 0 & 0 & 1 & \cdots & 0 & -c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{d-1} \end{bmatrix}. \quad (5)$$

We can view  $\Lambda_p$  as representing a linear map  $\mathbb{C}^d \rightarrow \mathbb{C}^d$  relative to the standard basis  $\{e_0, \dots, e_{d-1}\}$  of  $\mathbb{C}^d$ . The matrix  $\Lambda_p$  arises naturally in the following context: Suppose  $\lambda \in \mathbb{C}$  is an algebraic number with minimum (monic) polynomial  $p(x) \in \mathbb{Q}[x]$  as above. Then  $\mathbb{Q}[\lambda] = \mathbb{Q}(\lambda)$  is a  $d$ -dimensional

vector space over  $\mathbb{Q}$  with basis  $B = \{1, \lambda, \lambda^2, \dots, \lambda^{d-1}\}$ . Restricted to  $\mathbb{Q}[\lambda]$ , multiplication by  $\lambda$  corresponds to the  $\mathbb{Q}$ -linear map  $\mathbb{Q}[\lambda] \rightarrow \mathbb{Q}[\lambda]$  represented in the basis  $B$  by the matrix  $\Lambda_p$ .

The next lemma is standard.

**Lemma 76.** *Let  $p(x)$  and  $\Lambda_p$  be as in Definition 75 where  $p$  has degree  $d > 0$ . Let  $\mu_0, \dots, \mu_{d-1} \in \mathbb{C}$  be the (not necessarily distinct) roots of  $p$ . Let*

$$V := V(\vec{\mu}) := \begin{bmatrix} 1 & \mu_0 & \mu_0^2 & \cdots & \mu_0^{d-1} \\ 1 & \mu_1 & \mu_1^2 & \cdots & \mu_1^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mu_{d-1} & \mu_{d-1}^2 & \cdots & \mu_{d-1}^{d-1} \end{bmatrix} \quad (6)$$

be the  $d \times d$  Vandermonde matrix with respect to  $\vec{\mu} := (\mu_0, \dots, \mu_{d-1})$  (that is,  $[V]_{ij} = \mu_i^j$  for all  $i, j \in \{0, \dots, d-1\}$ ), and let  $L$  be the  $d \times d$  diagonal matrix with diagonal entries  $[L]_{ii} := \mu_i$  for all  $i \in \{0, \dots, d-1\}$ . Then

$$V\Lambda_p = LV.$$

*Proof.* For any  $i, k \in \{0, \dots, d-1\}$ , the  $(i, k)$ th entry of  $V\Lambda_p$  is given by

$$[V\Lambda_p]_{ik} = \sum_{j=0}^{d-1} [V]_{ij} [\Lambda_p]_{jk} = \sum_{j=0}^{d-1} \mu_i^j [\Lambda_p]_{jk} = \begin{cases} \mu_i^{k+1} & \text{if } k < d-1, \\ -\sum_{j=0}^{d-1} c_j \mu_i^j & \text{if } k = d-1. \end{cases}$$

Since  $\mu_i$  is a root of  $p$ , we have  $-\sum_{j=0}^{d-1} c_j \mu_i^j = \mu_i^d$ , and so in either case we get  $[V\Lambda_p]_{ik} = \mu_i^{k+1} = \mu_i \mu_i^k = [L]_{ii} [V]_{ik} = [LV]_{ik}$ .  $\square$

**Remark.** If  $V$  is invertible (which occurs when all the  $\mu_i$  are pairwise distinct), then Lemma 76 says that the columns of  $V^{-1}$  are eigenvectors of  $\Lambda_p$  with respective eigenvalues  $\mu_0, \dots, \mu_{d-1}$ .  $\square$

**Lemma 77.** *Let  $D$  be any subring of  $\mathbb{C}$ , and let  $p(x)$  be a monic polynomial in  $D[x]$  of degree  $d > 0$  with roots  $\mu_0, \dots, \mu_{d-1} \in \mathbb{C}$ . Fix any  $\lambda \in \{\mu_0, \dots, \mu_{d-1}\}$  and any  $x \in \mathbb{C}$ . If  $x$  is in  $Q_\lambda$ , then there exist  $a_0, \dots, a_{d-1} \in D$  such that*

$$1. \ x = \sum_{j=0}^{d-1} a_j \lambda^j \text{ and}$$

$$2. \text{ there exists } P \in Q_{[x]} \text{ such that, for all } 0 \leq i < d, \sum_{j=0}^{d-1} a_j \mu_i^j = P(\mu_i);$$

further, the converse holds provided the  $\mu_i$  are all pairwise distinct.

Suppose  $a_0, \dots, a_{d-1} \in D$  satisfy (2.), above. Then for any  $0 \leq i < d$ , if  $\mu_i \in [0, 1]$ , then  $\sum_{j=0}^{d-1} a_j \mu_i^j \in [0, 1]$ .

*Proof.* We write  $p(x) = x^d + \sum_{i=0}^{d-1} c_i x^i$  for some  $c_0, \dots, c_{d-1} \in D$ . Let  $\Lambda := \Lambda_p : \mathbb{C}^d \rightarrow \mathbb{C}^d$  be the linear map given by the matrix of Equation (5), above, and let  $V := V(\vec{\mu})$  be the  $d \times d$  Vandermonde matrix given by Equation (6). Note that  $V e_0 = \mathbf{1}$ , where we define  $\mathbf{1} := (1, 1, \dots, 1)^\top$  to be the column vector of all 1's. Let  $L$  be the diagonal matrix of Lemma 76. That lemma gives  $V\Lambda = LV$ , so applying Lemma 73 with  $S := \{0, e_0\}$ , we have

$$V(Q_\Lambda(\{0, e_0\})) = Q_L(V(\{0, e_0\})) = Q_L(\{V(0), V(e_0)\}) = Q_L(\{0, \mathbf{1}\}), \quad (7)$$

and using Equation (4) of Lemma 74, we get

$$Q_L(\{0, 1\}) = \left\{ \sum_{i=0}^{d-1} \rho_{0,1}(P(\mu_i))e_i \mid P \in Q_{[x]} \right\} = \left\{ \sum_{i=0}^{d-1} P(\mu_i)e_i \mid P \in Q_{[x]} \right\} = \{P(\vec{\mu}) \mid P \in Q_{[x]}\} , \quad (8)$$

where we use  $P(\vec{\mu})$  as an abbreviation for the vector  $\sum_{i=0}^{d-1} P(\mu_i)e_i$ . Combining Equations (7) and (8) and applying  $e_k^\top$  to both sides for any  $0 \leq k < d$ , we get

$$\begin{aligned} e_k^\top V(Q_\Lambda(\{0, e_0\})) &= e_k^\top Q_L(\{0, 1\}) = e_k^\top \{P(\vec{\mu}) \mid P \in Q_{[x]}\} = \{e_k^\top P(\vec{\mu}) \mid P \in Q_{[x]}\} \\ &= \{P(\mu_k) \mid P \in Q_{[x]}\} = Q_{\mu_k} , \end{aligned}$$

the last equality by Lemma 53. In particular, if  $\lambda = \mu_k$ , then  $e_k^\top V(Q_\Lambda(\{0, e_0\})) = Q_\lambda$ , which gives us, for all  $x \in \mathbb{C}$ ,

$$x \in Q_\lambda \iff x \in e_k^\top V(Q_\Lambda(\{0, e_0\})) \iff (\exists a \in Q_\Lambda(\{0, e_0\}))[x = e_k^\top Va] . \quad (9)$$

Now note that  $\Lambda \in D^{d \times d}$  and that both 0 and  $e_0$  have all coordinates in  $\mathbb{Z} \subseteq D$ . From this it clearly follows that every vector in  $Q_\Lambda(\{0, e_0\})$  has all components in  $D$ , i.e.,  $Q_\Lambda(\{0, e_0\}) \subseteq D^d$ . Now let  $a = (a_0, \dots, a_{d-1})^\top \in \mathbb{C}^d$  be any vector. Then using (7) and (8) again, we have

$$\begin{aligned} a \in Q_\Lambda(\{0, e_0\}) &\implies a \in D^d \ \& \ Va \in V(Q_\Lambda(\{0, e_0\})) \iff a \in D^d \ \& \ Va \in \{P(\vec{\mu}) \mid P \in Q_{[x]}\} \\ &\iff a \in D^d \ \& \ (\exists P \in Q_{[x]})[Va = P(\vec{\mu})] , \end{aligned}$$

and the first implication is reversible provided  $V$  is invertible, i.e., provided  $\mu_0, \dots, \mu_{d-1}$  are pairwise distinct. Combining this with (9) yields

$$x \in Q_\lambda \implies (\exists a \in D^d) \left[ x = e_k^\top Va \ \& \ (\exists P \in Q_{[x]})[Va = P(\vec{\mu})] \right]$$

with the converse holding if  $\mu_0, \dots, \mu_{d-1}$  are pairwise distinct. This is just a restatement in vector form of the first part of the lemma where  $\lambda = \mu_k$ .

For the second part of the lemma, suppose  $\mu_i \in [0, 1]$ . Then we know that  $Q_{\mu_i} \subseteq [0, 1]$ . If  $a \in D^d$  is such that  $Va = P(\vec{\mu})$  for some  $P \in Q_{[x]}$ , then  $\sum_{j=0}^{d-1} a_j \mu_i^j = e_i^\top Va = e_i^\top P(\vec{\mu}) = P(\mu_i) \in Q_{\mu_i} \subseteq [0, 1]$ .  $\square$

Recall (Lemma 60) that if  $D$  is a discrete subring of  $\mathbb{C}$ , then any two distinct elements of  $D$  are at least unit distance apart, and consequently,  $D$  is (topologically) closed and  $D \cap \mathbb{R} = \mathbb{Z}$ . We now come to the proof of the main result of this section.

*Proof of Theorem 69.* Set  $\mu_{d-1} := \lambda$  for convenience, and write  $p(x) := x^d + \sum_{i=0}^{d-1} c_i x^i$ , where each  $c_i \in D$ . Let us start with a generalized assumption that  $0 \leq \mu_i \leq 1$  for all  $0 \leq i \leq k-1$ , where  $k \in \{1, \dots, d-1\}$  (we need  $k = d-1$  and  $k = d-2$  in the two cases described in the theorem). In case (1) when  $k = d-1$ , it follows that  $\lambda \notin ]0, 1[$ , because otherwise, we would have  $0 < \lambda < 1$  and thus  $0 < \lambda \prod_{i=0}^{d-2} \mu_i < 1$ , but this product is equal to  $\pm c_0$ , which cannot be in  $]0, 1[$ , because  $D$  is discrete. In case (2), it is given that  $\lambda \notin \mathbb{R}$ , so again  $\lambda \notin ]0, 1[$ . If  $\lambda \in D$ , then  $R_\lambda \subseteq D$  and is clearly discrete, so from now on, we can assume that  $\lambda \notin D$ . It follows that  $d \geq 2$  and that  $\lambda \notin [0, 1]$ , which in turn implies that  $Q_\lambda$  is unbounded (Corollary 39). In case (2), we may also assume that  $d \geq 3$ , for if  $d = 2$ , then  $Q_\lambda = R_\lambda \subseteq \mathbb{Z}[\lambda] = \mathbb{Z} + \lambda\mathbb{Z}$ , the latter being discrete (and closed).

We now can apply Lemma 77 directly. Let  $V := V(\vec{\mu})$  be the  $d \times d$  Vandermonde matrix such that  $[V]_{ij} = \mu_i^j$  for all legal indices  $i$  and  $j$ . Fix some  $x \in Q_\lambda$ . The first part of Lemma 77 then says that  $x$  is of the form

$$x = e_{d-1}^\top V a = (1, \lambda, \lambda^2, \dots, \lambda^{d-1}) a$$

for some  $a = (a_0, \dots, a_{d-1})^\top \in D^d$  such that  $V a = P(\vec{\mu})$ , where  $P(\vec{\mu})$  is as in the proof of Lemma 77. The second part of Lemma 77 says that  $V a \in [0, 1]^{\times k} \times \mathbb{C}^{(d-k)}$ .

Let  $W$  be the  $k \times k$  submatrix of  $V$  consisting of the first  $k$  rows and  $k$  columns, and let  $\vec{\mu}^j$  denote the  $k$ -dimensional column vector  $(\mu_0^j, \dots, \mu_{k-1}^j)^\top$  for  $k \leq j \leq d-1$ , that is, the first  $k$  entries of the  $j^{\text{th}}$  column of  $V$ . Let  $a_F := (a_0, \dots, a_{k-1})^\top \in D^k$  be the first  $k$  coordinates of  $a$ , and let  $a_L := (a_k, \dots, a_{d-1})^\top$  be the last  $d-k$  coordinates of  $a$ . Then the constraint  $V a \subseteq [0, 1]^{\times k} \times \mathbb{C}^{(d-k)}$  is equivalent to

$$W a_F + \sum_{j=k}^{d-1} a_j \vec{\mu}^j \subseteq [0, 1]^{\times k}. \quad (10)$$

Note that  $W = V(\mu_0, \dots, \mu_{k-1})$  is an invertible, real matrix, and so  $W^{-1}$  is a real matrix. Applying  $W^{-1}$  to both sides of (10) and subtracting, we get

$$a_F \in D^k \cap \left( \Omega - \sum_{j=k}^{d-1} a_j w_j \right), \quad (11)$$

where  $\Omega := W^{-1}([0, 1]^{\times k})$  and  $w_j := W^{-1} \vec{\mu}^j \in \mathbb{R}^k$ . Clearly,  $\Omega \subseteq \mathbb{R}^k$  is a bounded, convex parallelepiped and depends only on  $\mu_0, \dots, \mu_{k-1}$ . The expression in parentheses on the right-hand side of (11) is just a translation of  $\Omega$ . All this establishes that

$$Q_\lambda \subseteq \bigcup_{a_L \in D^{d-k}} \left\{ \sum_{j=k}^{d-1} a_j \lambda^j + \sum_{j=0}^k a_j \lambda^j \mid (a_0, \dots, a_{k-1})^\top \in D^k \cap \left( \Omega - \sum_{j=k}^{d-1} a_j w_j \right) \right\}. \quad (12)$$

Let  $W_L := (w_k, \dots, w_{d-1})$  be the  $k \times (d-k)$  matrix whose columns are the  $w_j$ 's. For any  $a_L \in D^{d-k}$ , define  $\Omega_{a_L} := (D^k \cap (\Omega - W_L a_L)) + W_L a_L = (D^k \cap (\Omega - \sum_{j=k}^{d-1} a_j w_j)) + \sum_{j=k}^{d-1} a_j w_j$ . Then clearly,  $\Omega_{a_L} \subseteq \Omega$ . Let  $b := (b_0, \dots, b_{k-1})^\top := a_F + W_L a_L = a_F + \sum_{j=k}^{d-1} a_j w_j$ . Then, (12) becomes

$$\begin{aligned} Q_\lambda &\subseteq \bigcup_{a_L \in D^{d-k}} \left\{ (\lambda^k, \dots, \lambda^{d-1}) a_L + (\lambda^0, \dots, \lambda^{k-1}) a_F \mid a_F \in D^k \cap (\Omega - W_L a_L) \right\} \\ &= \bigcup_{a_L \in D^{d-k}} \left\{ (\lambda^k, \dots, \lambda^{d-1}) a_L + (\lambda^0, \dots, \lambda^{k-1}) (b - W_L a_L) \mid b \in \Omega_{a_L} \right\} \\ &= \bigcup_{a_L \in D^{d-k}} \left\{ ((\lambda^k, \dots, \lambda^{d-1}) - (\lambda^0, \dots, \lambda^{k-1}) W_L) a_L + (\lambda^0, \dots, \lambda^{k-1}) b \mid b \in \Omega_{a_L} \right\} \\ &= \bigcup_{a_L \in D^{d-k}} \left[ ((\lambda^k, \dots, \lambda^{d-1}) - (\lambda^0, \dots, \lambda^{k-1}) W_L) a_L + \left\{ (\lambda^0, \dots, \lambda^{k-1}) b \mid b \in \Omega_{a_L} \right\} \right]. \end{aligned}$$

The set  $S_{a_L} := \{(\lambda^0, \dots, \lambda^{k-1}) b \mid b \in \Omega_{a_L}\}$  in the right-hand side is clearly bounded, independent of  $a_L$ , because  $\Omega_{a_L} \subseteq \Omega$ . Set  $\Delta := (\lambda^k, \dots, \lambda^{d-1}) - (\lambda^0, \dots, \lambda^{k-1}) W_L$ . With these definitions,



we have

$$Q_\lambda \subseteq \bigcup_{a_L \in D^{d-k}} (\Delta a_L + S_{a_L}) .$$

Suppose that the set  $A := \bigcup_{a_L \in D^{d-k}} \{\Delta a_L\}$  has no accumulation points in  $\mathbb{C}$ , and moreover, that for any point  $p \in A$ , there are only finitely many values of  $a_L \in D^{d-k}$  such that  $p = \Delta a_L$ . Then, in any bounded region  $T \subseteq \mathbb{C}$  there are only finitely many  $a_L \in D^{d-k}$  such that  $T \cap (\Delta a_L + S_{a_L}) \neq \emptyset$ . Further,  $\Omega_{a_L}$  is a translated subset of  $D^k$ , and (owing to the discreteness of  $D$ ) there is a finite bound  $c$ , independent of  $a_L$ , on the cardinality of  $\Omega_{a_L}$  and thus of  $S_{a_L}$ . These facts together imply that there are only finitely many elements of  $Q_\lambda$  in any bounded region of  $\mathbb{C}$ . Thus  $Q_\lambda$  is discrete. Whence  $Q_\lambda = R_\lambda$ , implying that  $R_\lambda$  is discrete.

Now, we need to see when the set  $A = \bigcup_{a_L \in D^{d-k}} \{\Delta a_L\}$  satisfies the two assumptions above. We look at the two cases given in the theorem.

1. When  $k = d - 1$ , the set  $A = \bigcup_{a_{d-1} \in D} \{\Delta a_{d-1}\} = \Delta D$  (where  $\Delta = \lambda^{d-1} - (\lambda^0, \dots, \lambda^{d-2})w_{d-1}$ )

is clearly discrete. Also, we must have  $\Delta \neq 0$ , for otherwise,  $Q_\lambda \subseteq \bigcup_{a_{d-1} \in D} S_{a_{d-1}}$ , which is bounded, but we know that  $Q_\lambda$  is unbounded. Thus, for any point  $p \in A$  there is exactly one value of  $a_{d-1} \in D$  such that  $p = \Delta a_{d-1}$ .

2. When  $k = d - 2$ , we get  $A = \bigcup_{(a_{d-1}, a_{d-2}) \in D^2} \{a_{d-1}\alpha + a_{d-2}\beta\} = \alpha D + \beta D$ , where  $\alpha :=$

$\lambda^{d-1} - (\lambda^0, \dots, \lambda^{d-3})w_{d-1}$  and  $\beta := \lambda^{d-2} - (\lambda^0, \dots, \lambda^{d-3})w_{d-2}$ . It is easy to see that if  $D = \mathbb{Z}$  and moreover  $\beta \neq 0$  and  $\alpha/\beta \notin \mathbb{R}$  (i.e.,  $\alpha$  and  $\beta$  are in different directions), then the set  $A$  is discrete, and also, for any  $p \in A$  there is exactly one pair  $(a_{d-1}, a_{d-2}) \in \mathbb{Z}^2$  such that  $p = a_{d-1}\alpha + a_{d-2}\beta$ . Now, we show that  $\beta \neq 0$  and  $\alpha/\beta \notin \mathbb{R}$ .

By replacing  $w_{d-1}$  and  $w_{d-2}$  by their definitions, we can write

$$\alpha = \lambda^{d-1} - (\lambda^0, \dots, \lambda^{d-3})W^{-1}\vec{\mu}^{d-1} \quad \text{and} \quad \beta = \lambda^{d-2} - (\lambda^0, \dots, \lambda^{d-3})W^{-1}\vec{\mu}^{d-2} .$$

We know that  $WW^{-1} = I$ , so for any  $0 \leq i \leq d-3$ ,  $(\mu_i^0, \mu_i^1, \dots, \mu_i^{d-3})W^{-1} = e_i^\top$ . And hence,

$$(\mu_i^0, \mu_i^1, \dots, \mu_i^{d-3})W^{-1}\vec{\mu}^{d-2} = \mu_i^{d-2} . \tag{13}$$

Let us now look at  $\alpha$  and  $\beta$  as polynomials in  $\lambda$ . With some abuse of notation let us define the polynomial  $\beta(x) := x^{d-2} - (x^0, x^1, \dots, x^{d-3})W^{-1}\vec{\mu}^{d-2}$ . Equation (13) tells us that  $\beta(x)$  has  $\mu_0, \mu_1, \dots, \mu_{d-3}$  as its roots. As its degree is  $d-2$ , we can write

$$\beta(x) = (x - \mu_0)(x - \mu_1) \cdots (x - \mu_{d-3}) .$$

Similarly,  $\mu_0, \dots, \mu_{d-3}$  are also roots of the polynomial  $\alpha(x) := x^{d-1} - (x^0, \dots, x^{d-3})W^{-1}\vec{\mu}^{d-1}$ . But  $\alpha(x)$  has degree  $d-1$ , so it has one more root. The sum of all its roots is equal to minus the coefficient on  $x^{d-1}$ , which is zero, and so the other root is  $(-\mu_0 - \mu_1 \dots - \mu_{d-3})$ . Hence we can write the following:

$$\beta = \beta(\lambda) = (\lambda - \mu_0)(\lambda - \mu_1) \cdots (\lambda - \mu_{d-3})$$

and

$$\alpha = \alpha(\lambda) = (\lambda - \mu_0)(\lambda - \mu_1) \cdots (\lambda - \mu_{d-3})(\lambda + \mu_0 + \dots + \mu_{d-3}) .$$

Now it clear that  $\beta \neq 0$  and  $\alpha/\beta = \lambda + \mu_0 + \dots + \mu_{d-3}$ , which is non-real if and only if  $\lambda$  is non-real.

□

The proof of Theorem 69 actually establishes the following fact (cf. Equation (12)):

**Fact 78.** *Let  $D \subseteq \mathbb{C}$  be a discrete subring of  $\mathbb{C}$ , and let  $\lambda \in \mathbb{C}$  be a root of some monic polynomial  $p \in D[x]$  of degree  $d > 0$  such that  $p$  has no multiple roots in  $\mathbb{C}$ . Let  $\mu_0, \dots, \mu_{d-2} \in \mathbb{C}$  be the other roots of  $p$  besides  $\lambda$ , and assume that  $\mu_0, \dots, \mu_{k-1}$  are all elements of  $[0, 1]$ , for some  $k < d$ . Then letting*

- $W := V(\mu_0, \dots, \mu_{k-1})$ ,
- $\vec{\mu}^j := (\mu_0^j, \dots, \mu_{k-1}^j)^\top$  for  $k \leq j < d$ ,
- $w_j := W^{-1}\vec{\mu}^j \in \mathbb{R}^k$  for  $k \leq j < d$ , and
- $\Omega := W^{-1}([0, 1]^{\times k})$ ,

we have

$$Q_\lambda \subseteq \left\{ \sum_{j=0}^{d-1} a_j \lambda^j \mid (a_0, \dots, a_{d-1})^\top \in D^d \text{ \& } (a_0, \dots, a_{k-1})^\top \in \Omega - \sum_{j=k}^{d-1} a_j w_j \right\}. \quad (14)$$

The next corollary exhaustively applies case (1) of Theorem 69 with  $D = \mathbb{Z}$  and  $d = 2$ .

**Corollary 79.** *Let  $m$  and  $n$  be integers such that  $0 \leq n \leq 1 - m$ . Let  $\lambda = (m - \sqrt{m^2 + 4n})/2$ . Then  $R_\lambda$  is discrete, and if  $m^2 + 4n \neq 0$ , then*

$$R_\lambda \subseteq \{a - b\lambda \mid a, b \in \mathbb{Z} \text{ \& } b\lambda_+ \leq a \leq b\lambda_+ + 1\}, \quad (15)$$

where  $\lambda_+ := (m + \sqrt{m^2 + 4n})/2$ . If  $0 < \lambda_+ < 1$ , then  $\lambda < -1$ , and except for 0 and 1, any two distinct elements of  $R_\lambda$  differ by at least  $-\lambda$ .

*Proof.* Note that  $\lambda$  and  $\lambda_+$  are the roots of the quadratic polynomial  $p(x) := x^2 - mx - n$ . The inequality  $0 \leq n \leq 1 - m$  is equivalent to  $0 \leq \lambda_+ \leq 1$ . If  $\lambda = \lambda_+$ , then it follows that  $\lambda \in \mathbb{Z}$ , and so  $R_\lambda \subseteq \mathbb{Z}$  and we're done. If  $m^2 + 4n \neq 0$ , then  $\lambda \neq \lambda_+$ , and so Theorem 69 (case (1)) with  $D := \mathbb{Z}$  and  $(\mu_0, \mu_1) := (\lambda_+, \lambda)$  says that  $R_\lambda$  is discrete (and thus  $R_\lambda = Q_\lambda$ ). Applying Fact 78 with:  $d := 2$ ;  $k := 1$ ;  $(\mu_0, \mu_1) := (\lambda_+, \lambda)$ ;  $W := [1]$ ;  $\vec{\mu}^{d-1} = \vec{\mu}^1 := [\lambda_+]$ ;  $w_{d-1} := W^{-1}\vec{\mu}^1 = [\lambda_+]$ ; and  $\Omega := W^{-1}[0, 1] = [0, 1]$ , we have (using  $(a, b)$  instead of  $(a_0, a_1)$  as the index)

$$Q_\lambda \subseteq \{a + b\lambda \mid a, b \in \mathbb{Z} \text{ \& } a \in [0, 1] - b\lambda_+\}.$$

By switching  $b$  with  $-b$ , this set inclusion is seen to be equivalent to (15).

If  $0 < \lambda_+ < 1$ , then  $\lambda_+$  is irrational ( $\lambda_+$  is an algebraic integer), and it is easy to check that  $\lambda < -1$ . For any  $a, b \in \mathbb{Z}$ , if  $a - b\lambda \in R_\lambda$ , then  $b\lambda_+ \leq a \leq b\lambda_+ + 1$ , and if  $b \neq 0$ , there is exactly one  $a$  that satisfies this constraint; furthermore, this value of  $a$  ascends monotonically with  $b$ . If  $b = 0$ , then  $a \in \{0, 1\}$ , which corresponds to  $0, 1 \in R_\lambda$ . Otherwise, two elements of  $R_\lambda$  must differ by  $x - y\lambda$  for integers  $x, y$  such that  $x \geq 0$  and  $y > 0$ . Clearly,  $x - y\lambda \geq -\lambda$ . □

The case where  $m = -1$  and  $n = 1$  was already shown in Proposition 48. In that case,  $\lambda = -\varphi$  and  $R_\lambda = R_{-\varphi} = R_{1+\varphi}$ . Here are a few other cases:

$m = -2$  **and**  $n = 1$ :  $\lambda = -1 - \sqrt{2}$  and  $R_\lambda = R_{1-\lambda} = R_{2+\sqrt{2}}$ .

$m = -2$  **and**  $n = 2$ :  $\lambda = -1 - \sqrt{3}$  and  $R_\lambda = R_{1-\lambda} = R_{2+\sqrt{3}}$ .

$m = -3$  **and**  $n = 1$ :  $\lambda = -(3 + \sqrt{13})/2$  and  $R_\lambda = R_{1-\lambda} = R_{(5+\sqrt{13})/2}$ .

The next corollary exhaustively applies case (2) of Theorem 69 with  $d = 3$ .

**Corollary 80.** *Let  $a, b, c \in \mathbb{Z}$  be such that*

1.  $c < 0$ ,
2.  $a + b + c \geq 0$ , and
3.  $4b^3 + 36abc - 16a^3c + 27c^2 > 0$ .

*Let  $p(x) := x^3 + ax^2 + bx + c$ , let  $\mu$  be the (unique) root of  $p$  in  $]0, 1[$ , and let  $\lambda$  be one of the roots of  $p$  not in  $]0, 1[$ . Then  $R_\lambda$  is discrete, and moreover,*

$$\begin{aligned} R_\lambda &\subseteq \{a_0 + a_1\lambda + a_2\lambda^2 \mid a_0, a_1, a_2 \in \mathbb{Z} \ \& \ a_0 + a_1\mu + a_2\mu^2 \in [0, 1]\} \\ &= \{a_0 + a_1\lambda + a_2\lambda^2 \mid a_0, a_1, a_2 \in \mathbb{Z} \ \& \ -a_1\mu - a_2\mu^2 \leq a_0 \leq -a_1\mu - a_2\mu^2 + 1\} \\ &= \{1\} \cup \{a\lambda + b\lambda^2 + \lceil -a\mu - b\mu^2 \rceil \mid a, b \in \mathbb{Z}\}. \end{aligned}$$

*Proof sketch.* The first two conditions give  $p(0) = c < 0$  and  $p(1) = 1 + a + b + c > 0$ , respectively, and together they force  $p$  to have an odd number of roots in  $]0, 1[$ . Let  $r_1$  and  $r_2$  be the two roots of  $p'$  (the derivative of  $p$ ) in  $\mathbb{C}$ . One can show that  $p$  has three real roots if and only if  $p(r_1)p(r_2) \leq 0$ . A long and tedious calculation shows that

$$p(r_1)p(r_2) = \frac{1}{27}(4b^3 + 36abc - 16a^3c + 27c^2),$$

and thus the third condition means that the other two roots of  $p$ , including  $\lambda$ , are non-real (and hence distinct, because they are complex conjugates of each other). Thus the conditions of case (2) of Theorem 69 are satisfied, and  $R_\lambda$  is discrete. Now we apply Fact 78 with  $D := \mathbb{Z}$ , with  $d := 3$ , with  $\mu_0 := \mu$ , with  $k := 1$ , and with

- $W := V(\mu_0) = [1]$ ,
- $\vec{\mu}^j := [\mu^j]$  for  $j \in \{1, 2\}$ ,
- $w_j := [\mu^j]$  for  $j \in \{1, 2\}$ ,
- $\Omega := [0, 1]$

to get

$$R_\lambda = Q_\lambda \subseteq \{a_0 + a_1\lambda + a_2\lambda^2 \mid a_0, a_1, a_2 \in \mathbb{Z} \ \& \ a_0 \in [0, 1] - a_1\mu - a_2\mu^2\},$$

which is equivalent to the first containment in the corollary. The next equality is immediate. The final equality follows from the fact that  $p$ , having no integral roots, is irreducible, and thus the set  $\{1, \mu, \mu^2\}$  is linearly independent over  $\mathbb{Q}$ , whence  $-a\mu - b\mu^2 \notin \mathbb{Z}$  unless  $a = b = 0$ .  $\square$

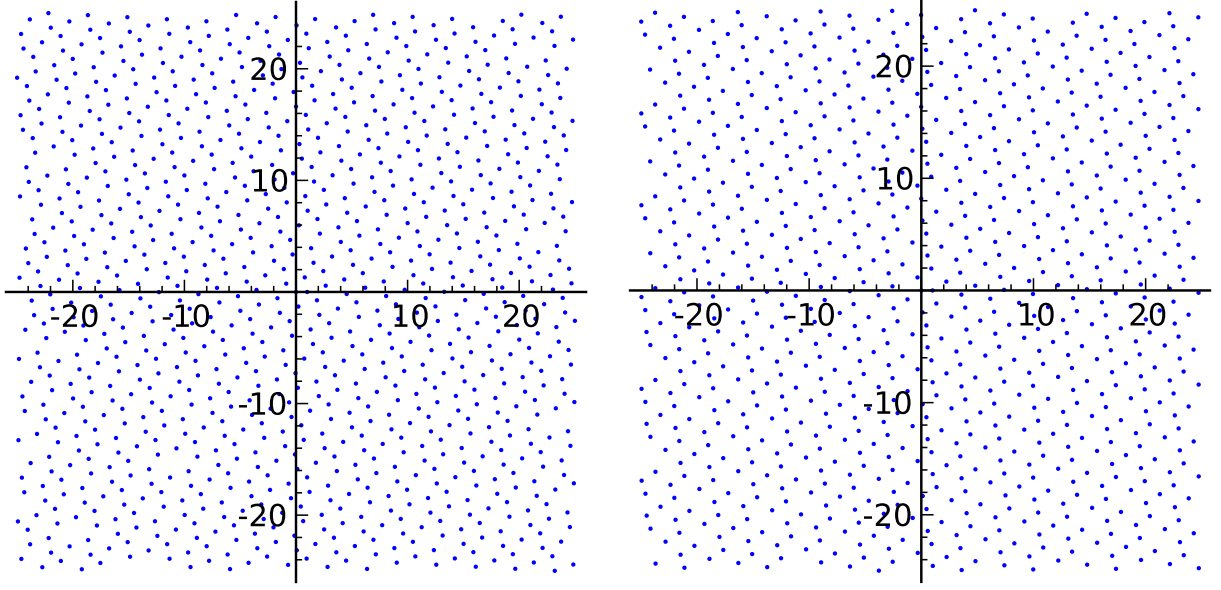


Figure 9: Two plots illustrating Corollary 80. The left plot is of  $R_\lambda$ , where  $\lambda$  is the root of the polynomial  $x^3 + x^2 - 1$  closest to  $-0.877 + 0.745i$ . The right plot is of  $R_\mu$ , where  $\mu$  is the root of the polynomial  $x^3 - x^2 + 2x - 1$  closest to  $-0.341 + 1.162i$ . One can easily show that  $R_\lambda = R_{\lambda^2} = R_{\lambda^3}$ .

Figure 9 shows two applications of Corollary 80.

Let  $r := \lambda - \mu$ , let  $s := \lambda^2 - \mu^2 = r(\lambda + \mu)$ , and consider the lattice  $r\mathbb{Z} + s\mathbb{Z}$ . This lattice is discrete, since  $r$  and  $s$  are  $\mathbb{R}$ -linearly independent, and it is interesting to notice that each point in  $R_\lambda$  differs from a point in this lattice by some real displacement between 0 and 1. It would be nice to know whether or when equality holds in Corollary 80.

## 11 Constraints on $Q_{[x]}$

In this section we prove a basic necessary condition for a polynomial  $P$  to be an element of  $Q_{[x]}$  (see Definition 50). This allows us to list all the polynomials in  $Q_{[x]}$  of degree  $\leq 2$  and get a finite upper bound on the number of polynomials in  $Q_{[x]}$  of any given degree bound.

Recall that every polynomial in  $Q_{[x]}$  has all integer coefficients.

**Lemma 81.** *Let  $P$  be any polynomial in  $Q_{[x]}$ .*

1.  $\{P(0), P(1)\} \subseteq \{0, 1\}$ .
2. If  $P$  is not constant, then  $0 < P(\lambda) < 1$  for all  $0 < \lambda < 1$ .

*Proof.* We have  $P(0) \in Q_0 = R_0 = \{0, 1\}$ . The first inclusion is by Lemma 53, and the last equality is by Fact 18. Likewise,  $P(1) \in Q_1 = R_1 = \{0, 1\}$ . This proves (1.).

Now suppose  $0 < \lambda < 1$ . We prove (2.) by induction on the minimum length  $\ell$  of a derivation of  $P$  (see Definition 51). Since  $P$  is not constant, we must have  $\ell \geq 3$ , and there exist  $S, T \in Q_{[x]}$ , both possessing derivations shorter than  $\ell$ , such that  $P(\lambda) = (1 - \lambda)S(\lambda) + \lambda T(\lambda)$ .  $S$  and  $T$  cannot be equal, for otherwise,  $P = S = T$ , violating the minimality of  $\ell$ . Therefore, either

- $S$  and  $T$  are both constant, in which case,  $S(\lambda) \neq T(\lambda)$ , and  $P(\lambda)$  is strictly between  $S(\lambda)$  and  $T(\lambda)$ , which are both in  $\{0, 1\}$  by (1.), or
- at least one of  $S$  and  $T$  is nonconstant, in which case, either  $0 < S(\lambda) < 1$  or  $0 < T(\lambda) < 1$ , by the inductive hypothesis.

In either case, we must have  $0 < P(\lambda) < 1$ . □

**Proposition 82.**

1. There are exactly four elements of  $Q_{[x]}$  of degree  $\leq 1$ , namely,

$$P_0 := 0, \quad P_1 := 1, \quad P_2 := x, \quad P_3 := 1 - x.$$

2. There are exactly ten elements of  $Q_{[x]}$  of degree 2, namely,

$$\begin{aligned} P_4 &:= x^2, & P_5 &:= -x^2 + 1, \\ P_6 &:= -x^2 + 2x, & P_7 &:= x^2 - 2x + 1, \\ P_8 &:= -x^2 + x, & P_9 &:= x^2 - x + 1, \\ P_{10} &:= -2x^2 + 2x, & P_{11} &:= 2x^2 - 2x + 1, \\ P_{12} &:= -3x^2 + 3x, & P_{13} &:= 3x^2 - 3x + 1. \end{aligned}$$

*Proof.* For (1.), we note that these are the only four polynomials  $P$  of degree  $\leq 1$  satisfying Lemma 81(1.), and each is easily seen to be in  $Q_{[x]}$ .

Any polynomial  $P$  of degree  $\leq 2$  is uniquely determined by its values on three distinct inputs. We consider  $P(0)$ ,  $P(1/2)$ , and  $P(1)$ . If, in addition,  $P \in Q_{[x]}$  and is nonconstant, then by Lemma 81, we have: (i)  $P(0) \in \{0, 1\}$ ; (ii)  $P(1) \in \{0, 1\}$ ; and (iii)  $0 < P(1/2) < 1$ . Since  $P \in \mathbb{Z}[x]$ ,  $P(1/2)$  is a multiple of  $1/4$ , and thus (iii) implies  $P(1/2) \in \{1/4, 1/2, 3/4\}$ . Taking all possible combinations, there are then at most  $2 \cdot 2 \cdot 3 = 12$  many  $P \in Q_{[x]}$  with degree 1 or 2 satisfying (i), (ii), and (iii). Two of these have degree 1 ( $P_2$  and  $P_3$ , above). The other ten have degree 2 and are listed above as  $P_4, \dots, P_{13}$ . We verify that they are all in  $Q_{[x]}$  by giving explicit derivations:

$$\begin{aligned} P_4 &= (1 - x)P_0 + xP_2, & P_5 &= (1 - x)P_1 + xP_3, \\ P_6 &= (1 - x)P_2 + xP_1, & P_7 &= (1 - x)P_3 + xP_0, \\ P_8 &= (1 - x)P_0 + xP_3, & P_9 &= (1 - x)P_1 + xP_2, \\ P_{10} &= (1 - x)P_2 + xP_3, & P_{11} &= (1 - x)P_3 + xP_2, \\ P_{12} &= (1 - x)P_6 + xP_5, & P_{13} &= (1 - x)P_7 + xP_4. \end{aligned}$$

□

We can use the same technique to get finite upper bounds on the number of elements of  $Q_{[x]}$  with any given degree. If the degree is at least four, then slightly better bounds can be obtained by using a classic theorem of Chebyshev [2] (see [1, Chapter 21]) to bound the leading coefficient by  $4^{n-1}$  in absolute value. We also can eliminate some polynomials from  $Q_{[x]}$  using the following fact:

**Fact 83.** *Let  $P \in \mathbb{R}[x]$  be any real polynomial satisfying condition (1.) of Lemma 81. Then  $P$  satisfies condition (2.) of the same lemma if and only if  $0 < P(r) < 1$  for every root  $r$  of  $P'$  (the derivative of  $P$ ) such that  $0 < r < 1$ .*

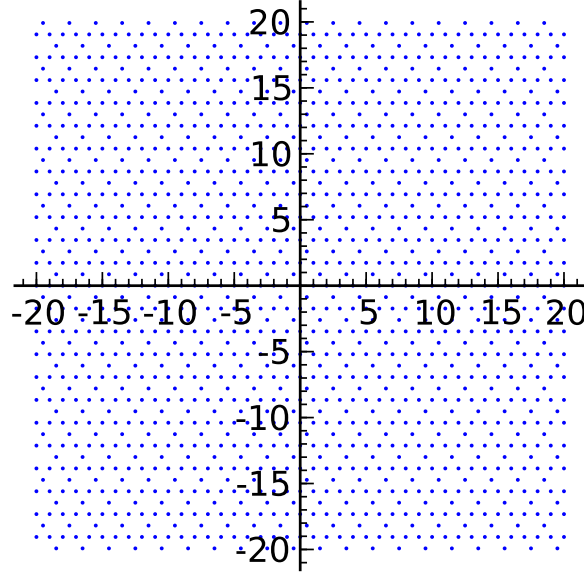


Figure 10:  $R_2(H)$ , the 2-clonvex closure of a regular hexagon. It is equal to  $\{a + b\eta \mid a, b \in \mathbb{Z} \text{ \& } ab \equiv 0 \pmod{2}\}$ , where  $\eta := e^{i\pi/3}$  is the principal third root of unity.

## 12 $\lambda$ -convex closures of some other finite sets

Here we briefly consider sets of the form  $Q_\lambda(S)$  and  $R_\lambda(S)$  where  $S \neq \{0, 1\}$ . We illustrate the  $\lambda$ -convex closures of three such sets: the 2-closure of a regular hexagon (Figure 10), the  $(1 + \varphi)$ -convex closure of a regular pentagon (Figure 11), and the  $(2 + \sqrt{2})$ -convex closure of a regular octagon (Figure 12). We will prove that all three of these sets are discrete.

If  $R_\lambda$  is convex, then so is  $R_\lambda(S)$  for any  $S$ , by Proposition 27. Thus to find discrete, nontrivially generated  $\lambda$ -clonvex sets, we must have  $R_\lambda$  discrete.

For example, let  $H$  be the set of vertices of a regular hexagon. For concreteness, assume the hexagon is positioned in the plane with its bottom side coinciding with  $[0, 1]$  on the real line. (We make the same positioning convention with all our regular polygons. Positioning this way conveniently ensures that  $\{0, 1\} \subseteq H$ .) If  $\lambda = 2$ , then  $R_2(H)$  is a proper subset of the Eisenstein integers (the regular triangular lattice) that has both translational symmetry (in six directions) and  $D_6$  rotational symmetry<sup>4</sup> about the center of the hexagon. For another example, if  $S = \{0, 1, i, 1+i\}$  is the set of vertices of a square (similarly positioned), then  $R_2(S) = \mathbb{Z}[i]$ , the set of Gaussian integers. If  $L = \{0, 1, i\}$ , then  $R_2(L)$  is a proper subset of  $\mathbb{Z}[i]$ , however, as the following lemma shows:

**Lemma 84.** *Let  $\gamma$  be any element of  $\mathbb{C} - \mathbb{Q}$ . Then*

$$Q_2(\{0, 1, \gamma\}) = \{a + b\gamma \mid a, b \in \mathbb{Z} \text{ \& } ab \equiv 0 \pmod{2}\}. \quad (16)$$

*If  $\gamma \in \mathbb{R}$ , then  $R_2(\{0, 1, \gamma\}) = \mathbb{R}$ ; otherwise,  $R_2(\{0, 1, \gamma\}) = Q_2(\{0, 1, \gamma\})$ , which is a discrete set.*

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<sup>4</sup> $D_n$  is the dihedral group of order  $2n$ .

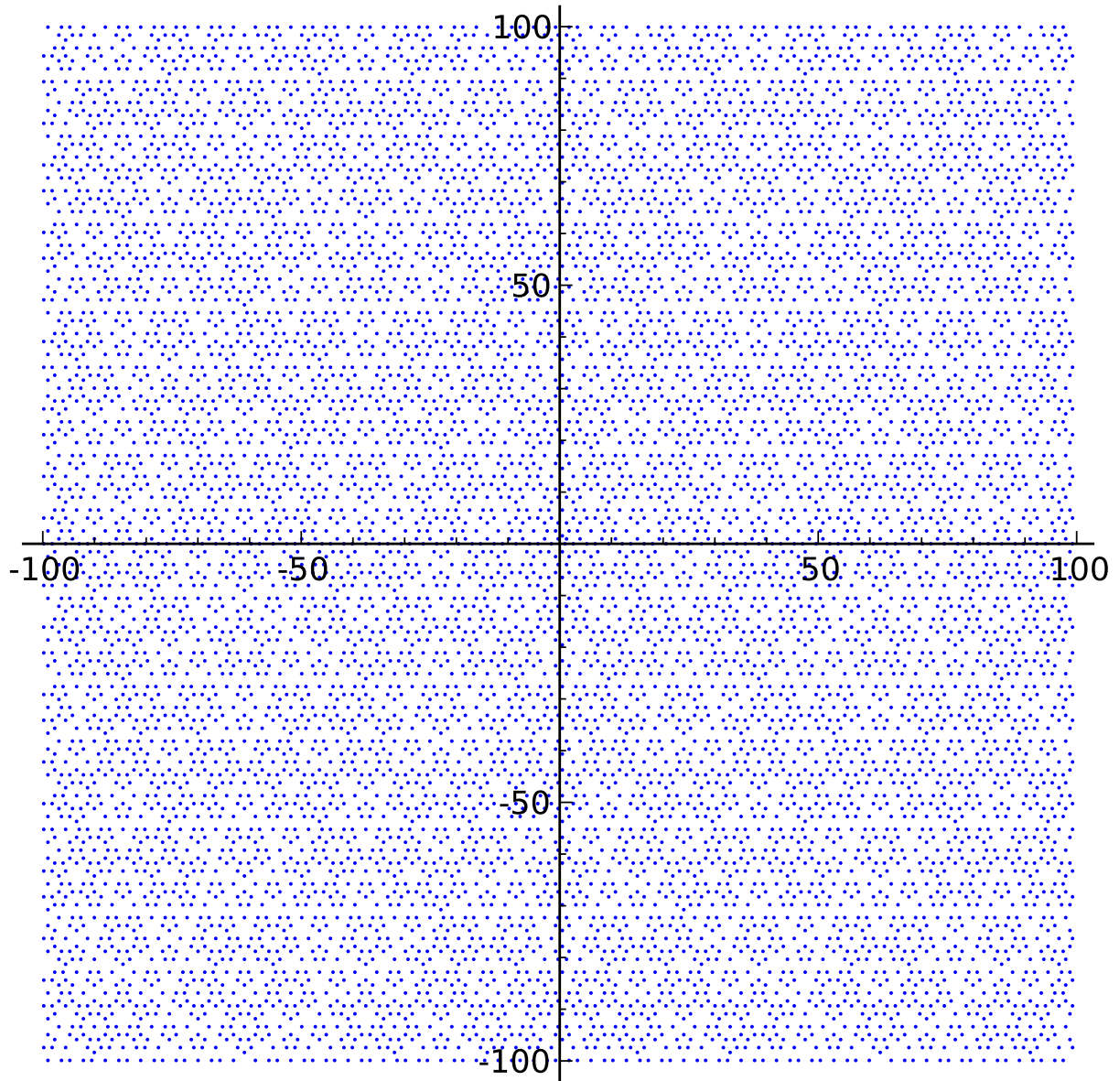


Figure 11:  $R_{1+\varphi}(P)$ , the  $(1 + \varphi)$ -convex closure of a regular pentagon. This set is discrete, but unlike  $R_2(H)$ , it has no translational symmetry.

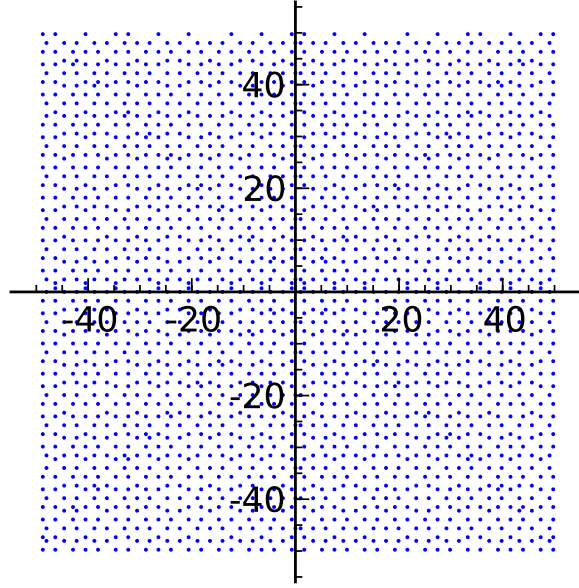


Figure 12:  $R_{2+\sqrt{2}}(O)$ , the  $(2 + \sqrt{2})$ -clonvex closure of a regular octagon. This set is discrete, with any two of its points at least unit distance apart.

*Proof.* It is easy to check, using the fact that  $\rho_{x,y}(2) = 2y - x$  for any  $x$  and  $y$ , that the right-hand side of (16) contains  $\{0, 1, \gamma\}$  and is 2-convex. This proves  $\subseteq$ .

For the reverse inclusion, we first note that  $Q_2(\{0, 1\}) = \mathbb{Z}$  and  $Q_2(\{0, \gamma\}) = \gamma\mathbb{Z}$ , and thus  $\mathbb{Z} \cup \gamma\mathbb{Z} \subseteq Q_2(\{0, 1, \gamma\})$ . Now let  $a, b \in \mathbb{Z}$  be arbitrary. If  $b$  is even, then  $a + b\gamma = \rho_{-a, b\gamma/2}(2)$ . If  $a$  is even, then  $a + b\gamma = \rho_{-b\gamma, a/2}(2)$ . In either case, this shows that  $a + b\gamma \in Q_2(\mathbb{Z} \cup \gamma\mathbb{Z}) = Q_2(\{0, 1, \gamma\})$ , which establishes the reverse containment.

If  $\gamma$  is a real, irrational number, then (16) implies that  $Q_2(\{0, 1, \gamma\})$  is a dense subset of  $\mathbb{R}$ , and so  $R_2(\{0, 1, \gamma\}) = \mathbb{R}$ . If  $\gamma \notin \mathbb{R}$ , then  $Q_2(\{0, 1, \gamma\})$  is a subset of the discrete lattice generated by 1 and  $\gamma$ .  $\square$

Considering  $R_2(H)$  again, one can easily check that  $H \subseteq R(\{0, 1, \eta\})$  where  $\eta := e^{i\pi/3}$  is the principal third root of unity. Thus, by Lemma 84,  $R_2(H) = R_2(\{0, 1, \eta\}) = \{a + b\eta \mid a, b \in \mathbb{Z} \text{ \& } ab \equiv 0 \pmod{2}\}$ , and this set is shown in Figure 10.

Let  $P$  be the vertices of a regular pentagon oriented in the plane as described above, so that the base of the pentagon coincides with  $[0, 1]$ . The set  $R_{1+\varphi}(P)$  is illustrated in Figure 11. Like  $R_2(H)$ , this set is discrete (as we show below) and has  $D_5$  symmetry about the center of  $P$ . Unlike  $R_2(H)$ , however,  $R_{1+\varphi}(P)$  has no translational symmetry.

**Theorem 85.**  $R_{1+\varphi}(P)$  is discrete; moreover, except for adjacent points in  $P$ , all its points are farther than unit distance apart.

*Proof.* We show discreteness directly first; it is implied by the stronger distance property, but readers may wish to skip the proof of the latter. Let  $S := Q_{1+\varphi}(P)$ . The idea of the proof is as follows: If we project each point in  $S$  orthogonally onto the imaginary axis, then the image is an



affine translation of  $Q_{1+\varphi}$ , which we already know is discrete. This means that the imaginary parts of the points in  $S$  are restricted to a discrete set, and so  $S$  is confined to a countable, discrete set of horizontal lines. By rotational symmetry, we can project the points in  $S$  orthogonally onto another axis, not parallel to the imaginary axis, obtaining another discrete set of images, confining  $S$  to another discrete set of parallel, nonhorizontal lines. The intersection of these two sets of lines is clearly discrete and closed, and so  $R_{1+\varphi}(P) = S$ , which is discrete.

Now the formal details. Recall that the  $\mathbb{R}$ -linear map  $\Im : \mathbb{C} \rightarrow \mathbb{R}$  maps each complex number to its imaginary part. Let  $\Lambda : \mathbb{C} \rightarrow \mathbb{C}$  be multiplication by  $1 + \varphi$ , regarded as an  $\mathbb{R}$ -linear map. Since  $\Lambda(\mathbb{R}) = \mathbb{R}$ , we also let  $\Lambda$  denote the restriction of this map to  $\mathbb{R}$ . Clearly,  $\Lambda \circ \Im = \Im \circ \Lambda$ , and so by Lemma 73 (with  $k := \mathbb{R}$ ,  $U := \mathbb{C}$ ,  $V := \mathbb{R}$ ,  $\Lambda_U := \Lambda_V := \Lambda$ , and  $t := \Im$ ), we have

$$\Im(S) = \Im(Q_{1+\varphi}(P)) = \Im(Q_\Lambda(P)) = Q_\Lambda(\Im(P)) = Q_{1+\varphi}(\Im(P)) .$$

A little elementary geometry shows that  $\Im(P) = \{0, \zeta, \kappa\}$ , where

$$\zeta := \sin(\tau/5) = \frac{\sqrt{2}}{4} \sqrt{5 + \sqrt{5}} \approx 0.951 , \quad \kappa := \varphi\zeta = \frac{\sqrt{5 + 2\sqrt{5}}}{2} \approx 1.539 .$$

From this we readily get  $0 = -\varphi\kappa + (1 + \varphi)\zeta = \rho_{\kappa, \zeta}(1 + \varphi) \in Q_{1+\varphi}(\{\kappa, \zeta\})$ , and therefore

$$Q_{1+\varphi}(\Im(P)) = Q_{1+\varphi}(\{0, \zeta, \kappa\}) = Q_{1+\varphi}(\{\kappa, \zeta\}) = \rho_{\kappa, \zeta}(Q_{1+\varphi}) .$$

The right-hand side is discrete by Proposition 48, which shows that  $\Im(S)$  is discrete.

Now define  $\Im'(z) := \Im(e^{i\tau/5}z)$  for all  $z \in \mathbb{C}$ . We also get  $\Im'(P) = \{0, \zeta, \kappa\}$ , and a similar argument using  $\Im'$  instead of  $\Im$  yields  $\Im'(S) = \rho_{\kappa, \zeta}(Q_{1+\varphi})$  and is therefore discrete. Combining these arguments gives  $S \subseteq \Im^{-1}(Q_{1+\varphi}) \cap (\Im')^{-1}(Q_{1+\varphi})$ , and the right-hand side is discrete and closed. This shows that  $R_{1+\varphi}(P) = S$  and is discrete.

Fix any  $a, b \in S$  such that  $0 < |b - a| \leq 1$ . We now show that  $|b - a| = 1$  and  $a, b \in P$ . Let  $C$  be the center of  $P$ , and let  $G$  be the group of rotations about  $C$  through angles that are multiples of  $\tau/5$ . ( $G$  is the 5-element cyclic group generated by  $\rho_{\xi, 0}$ , where  $\xi := e^{i3\tau/10}$ .) By symmetry, each element of  $G$  leaves  $P$ , and therefore  $S$ , invariant.<sup>5</sup>

**Claim 86.** *There exist distinct  $g_1, g_2 \in G$  such that*

1.  $\{\Im(g_1(a)), \Im(g_1(b))\} = \{\Im(g_2(a)), \Im(g_2(b))\} = \{\zeta, \kappa\}$ ,
2. *the slope of the line through  $g_1(a)$  and  $g_1(b)$  is  $\tan(\tau/10)$ , and*
3. *the slope of the line through  $g_2(a)$  and  $g_2(b)$  is  $-\tan(\tau/10)$ .*

*In particular,  $|b - a| = 1$ .*

*Proof of the Claim.* Let  $\theta$  be the argument of  $b - a$ , and let  $r = |b - a|$ . By assumption,  $0 < r \leq 1$ . The elements of  $G$  rotate the line segment connecting  $a$  with  $b$  to form line segments with length  $r$  and with arguments  $\theta + k\tau/5$ , for  $k \in \{0, 1, 2, 3, 4\}$ . The vertical displacement of each such line segment (that is, the absolute difference between the imaginary parts of the two endpoints) is thus  $r|\sin(\theta + k\tau/5)|$ . A simple geometric argument shows that there exist distinct  $k_1, k_2 \in \{0, 1, 2, 3, 4\}$

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<sup>5</sup>It is clear, in fact, that  $S$  has  $D_5$  symmetry about  $C$ .

such that  $0 < |\sin(\theta + k_1\tau/5)| \leq \sin(\tau/10)$  and  $0 < |\sin(\theta + k_2\tau/5)| \leq \sin(\tau/10)$ . Let  $g_1$  and  $g_2$  be the corresponding elements of  $G$ , respectively. Now we note that

$$\sin(\tau/10) = \frac{\sqrt{2}}{4} \sqrt{5 - \sqrt{5}} = \kappa - \zeta \approx 0.588.$$

We thus have

$$|\Im(g_1(b)) - \Im(g_1(a))| \leq \kappa - \zeta, \quad |\Im(g_2(b)) - \Im(g_2(a))| \leq \kappa - \zeta. \quad (17)$$

We know that  $g_1(a), g_1(b), g_2(a), g_2(b) \in S$ , and we established earlier that  $\Im(S) = \rho_{\kappa, \zeta}(Q_{1+\varphi})$ . Thus  $\Im(g_j(a))$  and  $\Im(g_j(b))$  are both in  $\rho_{\kappa, \zeta}(Q_{1+\varphi})$  for  $j \in \{1, 2\}$ . Now Proposition 48 implies that, except for  $\kappa$  and  $\zeta$ , any two adjacent points of  $\rho_{\kappa, \zeta}(Q_{1+\varphi})$  differ by at least  $(\kappa - \zeta)\varphi = \zeta$ , which is strictly greater than  $\kappa - \zeta$ . This establishes the first item of the Claim, and it also implies that equality must hold for both inequalities in (17). Therefore, for both  $j \in \{1, 2\}$ ,

$$\sin(\tau/10) = \kappa - \zeta = |\Im(g_j(b)) - \Im(g_j(a))| = r|\sin(\theta + k_j\tau/5)| \leq |\sin(\theta + k_j\tau/5)| \leq \sin(\tau/10).$$

The only way this can occur is when  $r = 1$  and  $|\sin(\theta + k_j\tau/5)| = \sin(\tau/10)$ . Since  $g_1 \neq g_2$ , it must then be that the line through  $g_1(a)$  and  $g_1(b)$  and the line through  $g_2(a)$  and  $g_2(b)$  have oppositely signed slopes, both with absolute value  $\tan(\tau/10)$ . By swapping  $g_1$  and  $g_2$  if necessary, we can assume that the line through  $g_1(a)$  and  $g_1(b)$  has positive slope. This establishes the rest of the Claim.  $\square$

It remains to show that  $a, b \in P$ , and for this it suffices that  $g_1(a)$  and  $g_1(b)$  are both in  $P$ . By swapping  $a$  and  $b$  if necessary, we can assume that  $\Im(g_1(a)) = \zeta$  and  $\Im(g_1(b)) = \kappa$ . To get the slope of the line between  $g_2(a)$  and  $g_2(b)$  to be  $-\tan(\tau/10)$ , as asserted by the Claim, it must be that  $g_2$  results from applying  $g_1$  followed by a clockwise rotation around  $C$  through angle  $\tau/5$ . Thus we have  $\Im(g_2(a)) = \kappa$  and  $\Im(g_2(b)) = \zeta$ , and this is only possible if  $g_1(b) = g_2(a)$  is the apex of  $P$  and  $g_1(a)$  is the element of  $P$  immediately to its left. This finishes the proof of Theorem 85.  $\square$

**Corollary 87.** *All  $\mathbb{R}$ -affine transformations of  $\mathbb{C}$  that are symmetries of  $R_{1+\varphi}(P)$  are length-preserving and leave  $P$  invariant.*

Let  $O$  be vertices of the regular octagon positioned in the plane with its bottom side coinciding with  $[0, 1]$ . The  $(2 + \sqrt{2})$ -clonvex closure of  $O$  is illustrated in Figure 12. As with  $R_{1+\varphi}(P)$ , this set is discrete and has no translational symmetry, although it has  $D_8$  symmetry about the center of  $O$ . The same techniques used to prove the following, whose proof we only sketch:

**Theorem 88.**  *$R_{2+\sqrt{2}}(O)$  is discrete; moreover, all points in  $R_{2+\sqrt{2}}(O)$  are at least unit distance apart.*

Unlike  $R_{1+\varphi}(P)$ ,  $R_{2+\sqrt{2}}(O)$  has infinitely many pairs of points that are unit distance apart; they radiate out from  $O$  in the eight directions whose angles are multiples of  $\tau/8$ , as can be seen in Figure 12.

*Proof sketch of Theorem 88.* Let  $S := R_{2+\sqrt{2}}(O)$ .  $\Im(S)$  is a subset of  $\rho_{\kappa, 1}(Q_{2+\sqrt{2}})$ , where this time,  $\kappa := \sqrt{2}/2$ . Similarly,  $\Re(S)$ —the set of real parts of elements of  $S$ —is a subset of  $\rho_{0, 1-\kappa}(Q_{2+\sqrt{2}})$ .  $Q_{2+\sqrt{2}}$  is discrete by Corollary 79, where  $m := -2$  and  $n := 1$ . This makes both the real and imaginary parts of elements of  $S$  drawn from discrete sets. Thus  $S$  is discrete.

By Corollary 79 with  $\lambda = 1 - (2 + \sqrt{2}) = -1 - \sqrt{2}$ , any two distinct elements of  $\Im(S)$  or of  $\Re(S)$  differ by at least  $(1 - \kappa)(-\lambda) = (2 - \sqrt{2})(1 + \sqrt{2})/2 = \sqrt{2}/2$ . It follows that if distinct  $x, y \in S$  have different real parts and different imaginary parts, then  $|x - y| \geq 1$ . If  $x$  and  $y$  have the same real part and different imaginary parts, or vice versa, then by symmetry we can rotate the plane about the center of  $O$  through angle  $\tau/8$  to obtain images  $x'$  and  $y'$ , both in  $S$ , whose real parts and imaginary parts both differ. Then we have  $|x - y| = |x' - y'| \geq 1$ .  $\square$

We conjecture that  $R_{2+\sqrt{2}}(O)$  has no  $\mathbb{R}$ -affine symmetries except those that leave  $O$  invariant.

### 13 The $\lambda$ -convex closure of a bent path

This section is dedicated to proving Theorem 34. We sequester the proof in its own section because it uses some concepts and techniques that are not used anywhere else in the paper, particularly, winding number and some basic homology and homotopy theory. We only need a few basic facts about these:

- For every loop  $\ell$  in  $\mathbb{C}$  and every point  $x$  not on  $\ell$ ,  $\ell$  has a well defined *winding number* about  $x$ , which is an integer indicating the number of times  $\ell$  “wraps around”  $x$ —positive for counterclockwise, negative for clockwise.
- If two loops are homologous in  $\mathbb{C} - \{x\}$  (that is, their difference can be expressed as the sum of boundaries of continuous images of disks in  $\mathbb{C} - \{x\}$ ), then they have the same winding number about  $x$ .
- Any two loops that are homotopic in  $\mathbb{C} - \{x\}$  are also homologous in  $\mathbb{C} - \{x\}$ , and thus have the same winding number about  $x$ .
- The winding number of a sum of loops—about some  $x$  not on any of the loops—is the sum of the winding numbers of the individual loops about  $x$ .
- If  $x, y \in \mathbb{C}$  and  $\ell$  is a loop in  $\mathbb{C} - \{x, y\}$  such that there is a path from  $x$  to  $y$  that does not intersect  $\ell$ , then  $\ell$  has the same winding number about  $y$  as it has about  $x$ .

**Definition 89.** A path in  $\mathbb{C}$  is *bent* if it does not lie within any single straight line.

Theorem 34 can then be restated as follows:

**Theorem 90.**  $Q_\lambda(c) = \mathbb{C}$  for any  $\lambda \in \mathbb{C} - [0, 1]$  and any bent path  $c$ .

**Definition 91.** Let  $c : [0, 1] \rightarrow \mathbb{C}$  be a path. A *subpath* of  $c$  is any path  $d : [0, 1] \rightarrow \mathbb{C}$  that starts at some point  $c(a)$ , follows  $c$ , and ends at some point  $c(b)$ , where  $0 \leq a \leq b \leq 1$ . That is, there exist  $0 \leq a \leq b \leq 1$  such that  $d(x) = c(\rho_{a,b}(x))$  for all  $x \in [0, 1]$ .

**Definition 92.** A path  $c$  is a *loop* iff it begins and ends in the same place, i.e.,  $c(0) = c(1)$ .

The proof of Theorem 90 uses the following lemma:

**Lemma 93.** If  $c : [0, 1] \rightarrow \mathbb{C}$  is a bent path that does not include any nonempty open subset of  $\mathbb{C}$ , then  $c$  includes a subpath  $d : [0, 1] \rightarrow \mathbb{C}$  with the following properties:

1.  $d$  lies entirely in a closed half-plane whose boundary passes through its endpoints  $d(0)$  and  $d(1)$ , and
2. there exists a point  $x \in \mathbb{C}$  such that the loop formed by first traversing  $d$  from  $d(0)$  to  $d(1)$  then following the straight line segment from  $d(1)$  back to  $d(0)$  has nonzero winding number about  $x$  (which is not on the loop).

*Proof.* Let  $A$ ,  $B$ , and  $C$  be three noncolinear points along  $c$ . We can assume without loss of generality that  $A = c(0)$ ,  $B = c(1/2)$ , and  $C = c(1)$  (otherwise, take an appropriately reparameterized subpath of  $c$ ). By our assumption about  $c$  not filling any space, we can choose a point  $x$  in the interior of the triangle  $\triangle ABC$  that does not lie on  $c$ . We first show that there is a subpath  $e$  of  $c$  that satisfies the second condition of the lemma with respect to  $x$ . Let  $t$  be the (oriented) loop formed by tracing the perimeter of  $\triangle ABC$ , starting at  $A$ , going to  $B$ , then to  $C$ , then back to  $A$ . Clearly,  $t$  has winding number  $\pm 1$  about  $x$ . (It is not necessary, but we can assume that  $t$  goes counterclockwise, so its winding number about  $x$  is  $+1$ .) Now,  $t$  is evidently homologous to the sum of the following three loops:

- the loop  $\ell_1$  obtained by first following the path  $c$  from  $A$  to  $C$  then the straight line segment from  $C$  back to  $A$ ,
- the loop  $\ell_2$  obtained by first following  $c$  backwards from  $B$  to  $A$  then the straight line segment from  $A$  back to  $B$ , and
- the loop  $\ell_3$  obtained by first following  $c$  backwards from  $C$  to  $B$  then the straight line segment from  $B$  back to  $C$ .

Since the winding number around  $x$  is invariant under homology of loops in  $\mathbb{C} - \{x\}$ , and the winding number of a sum is the sum of the winding numbers, it follows that the winding numbers (around  $x$ ) of  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  sum to 1. Thus at least one of these three loops has nonzero winding number around  $x$ . If it is  $\ell_1$ , then we take  $e := c$ ; if  $\ell_2$ , then we let  $e$  be  $c$  restricted to  $[0, 1/2]$  (reparameterized); and if it is  $\ell_3$ , then we take  $e$  to be  $c$  restricted to  $[1/2, 1]$  (reparameterized). Then  $e$  and  $x$  satisfy the second condition of the lemma (but not necessarily the first). See Figure 13. (Note that the orientation of  $e$  does not matter here, because the winding number will be nonzero regardless of orientation.)

We now find a subpath  $d$  of  $e$  that satisfies both conditions. Let  $x$  be as above (so that the loop that first follows  $e$  then goes straight back from  $e(1)$  to  $e(0)$  has nonzero winding number about  $x$ ), let  $L$  be the line through  $P := e(0)$  and  $Q := e(1)$ , and let  $L'$  be the line through  $x$  parallel to  $L$ . The situation might look like Figure 14.

Let  $H$  be the open halfplane of  $\mathbb{C}$  with boundary  $L$  and containing  $x$ . Then  $e^{-1}(H)$  is an open subset of  $]0, 1[$ , and hence is the disjoint union of at most countably many open intervals  $I_0, I_1, I_2, \dots \subseteq ]0, 1[$ . For  $i = 0, 1, 2, \dots$ , let  $e_i$  be the path  $e$  restricted to  $I_i$ . Now by the continuity of  $e$ , there can be only finitely many  $i$  such that  $e_i$  intersects  $L'$ . Indeed, suppose there were infinitely many such  $i$ , say  $i_0, i_1, i_2, \dots$ . For all  $j \in \{0, 1, 2, \dots\}$  let  $a_j$  be the left boundary of  $I_{i_j}$ , and let  $z_j \in I_{i_j}$  be such that  $e(z_j) \in L'$ . By the continuity of  $e$ , we must have  $e(a_j) \in L$ . The sequence  $z_0, z_1, z_2, \dots$  has some accumulation point  $z \in [0, 1]$ . Take some monotone subsequence  $z_{j_0}, z_{j_1}, z_{j_2}, \dots$  converging to  $z$ , where  $j_0 < j_1 < j_2 < \dots$ . If this sequence is increasing, then we have  $z_{j_k} < a_{j_{k+1}} < z_{j_{k+1}}$  for all  $k$ , and if it is decreasing, then we have  $z_{j_{k+1}} < a_{j_k} < z_{j_k}$  for all  $k$ . In either case, the sequence  $a_{j_0}, a_{j_1}, a_{j_2}, \dots$  also converges to  $z$ , but then since  $a_j \in e^{-1}(L)$  and

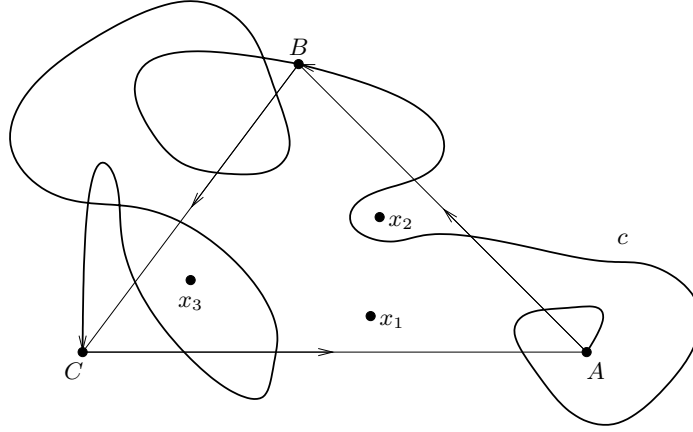


Figure 13: The curve  $c$  and the triangle  $\triangle ABC$ . For each  $i \in \{1, 2, 3\}$ , the loop  $\ell_i$  has nonzero winding number around the point  $x_i$ .

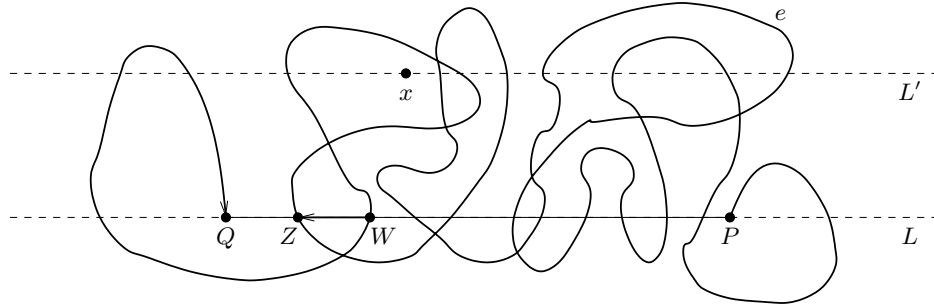


Figure 14: The curve  $e$  and the point  $x$ . (The small loop that follows the curve  $e$  from  $Z$  to  $W$  then straight back to  $Z$  has nonzero winding number around  $x$ .)

$z_j \in e^{-1}(L')$  for all  $j$ , and both  $e^{-1}(L)$  and  $e^{-1}(L')$  are closed, we have  $z \in e^{-1}(L) \cap e^{-1}(L') = \emptyset$ . Contradiction.

Now we have  $e_{i_1}, \dots, e_{i_n}$  intersect  $L'$  for some natural number  $n$  and some indices  $i_1, \dots, i_n$ , and no other  $e_i$  intersect  $L'$ . Let  $\ell$  be the loop that follows  $e$  forward from  $P$  to  $Q$  then follows  $L$  from  $Q$  back to  $P$ . For  $1 \leq j \leq n$ , let  $r_j < s_j$  be the boundaries of the interval  $I_{i_j}$ , and let  $\ell_j$  be the loop that follows  $e_{i_j}$  forward from  $e(r_j)$  to  $e(s_j)$  then  $L$  from  $e(s_j)$  to  $e(r_j)$  (note that  $e(r_j)$  and  $e(s_j)$  both lie on  $L$ ). We now can express  $\ell$  as the *finite* sum  $\ell_1 + \ell_2 + \dots + \ell_n + p$ , where  $p := \ell - (\ell_1 + \ell_2 + \dots + \ell_n)$ . This decomposition (for the curve shown in Figure 14) is illustrated in Figure 15.

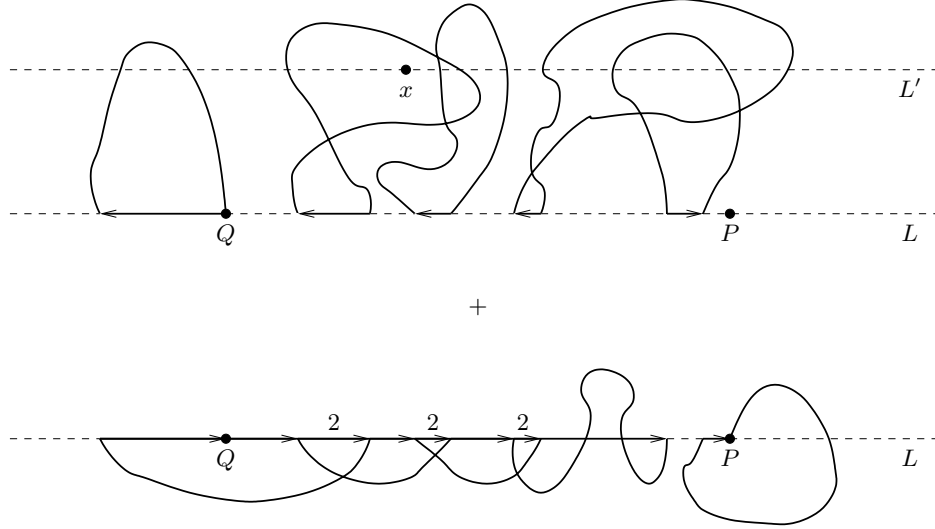


Figure 15: The decomposition of  $\ell$  in the previous figure as a sum  $\ell_1 + \dots + \ell_5$  of five loops that intersect  $L'$  (top) and those that don't ( $p$ , bottom). The “2” above a segment indicates that it counts double.

The winding number of  $\ell$  about  $x$  is nonzero, because  $e$  satisfies the second condition of the lemma with respect to  $x$ . This winding number is the sum of the winding numbers (about  $x$ ) of  $\ell_1, \dots, \ell_n$ , and  $p$ . There is no contribution from  $p$  to the winding number, because it stays entirely in the open halfplane bounded by  $L'$  and containing  $L$  (thus it cannot wrap around  $x$ ). It follows that there must be some  $\ell_j$  that has a nonzero winding number around  $x$  (e.g., the small loop through  $Z$  and  $W$  in Figure 14). Since  $e_{i_j}$  lies entirely in  $H$  (except for its endpoints), we can take  $d := e_{i_j}$ , which then satisfies both conditions of the lemma.  $\square$

We first prove the special case of Theorem 90 where  $\lambda$  is real. This is Proposition 94, below. Afterwards, we will explain how to modify the proof for nonreal  $\lambda$ .

**Proposition 94.**  $Q_\lambda(c) = \mathbb{C}$  for any  $\lambda \in \mathbb{R} - [0, 1]$  and any bent path  $c$ .

*Proof.* We can assume that  $\lambda > 1$  by Fact 9. If  $c$  includes a nonempty open subset of  $\mathbb{C}$ , then we are done by Proposition 33, and so from now on we assume that this is not the case. Then Lemma 93 implies that we can take a subpath  $d$  of  $c$  satisfying the two properties of the lemma with respect

to some point  $x$ , and then it is enough to show that  $Q_\lambda(d) = \mathbb{C}$ . And for *this* it is enough to show that  $Q_\lambda(d)$  contains a nonempty open subset of  $\mathbb{C}$ , thanks to Proposition 33.

The first property of Lemma 93 says that  $d$  lies entirely to one side of some line  $L$  through  $d(0)$  and  $d(1)$ . (If  $d(0) = d(1)$ , then  $L$  may not be unique.) Let  $\ell$  be the oriented loop that first follows  $d$  from  $d(0)$  to  $d(1)$ , then returns from  $d(1)$  back to  $d(0)$  along  $L$ . The second property of the lemma says that  $\ell$  has nonzero winding number about  $x$ .

Since  $d$  is compact and hence closed, there is some ball  $B$  of radius  $\varepsilon > 0$  about  $x$  that is disjoint from  $d \cup L$ . Furthermore,  $\ell$  has the same nonzero winding number about every point  $y \in B$  as it has about  $x$ . Now notice that the set  $B' := \{\rho_{y,d(0)}(\lambda) \mid y \in B\}$  is an open neighborhood of the point  $x' := \rho_{x,d(0)}(\lambda)$  (in fact, a ball centered at  $x'$  with radius  $\varepsilon(\lambda - 1)$ ). Figure 16 shows a typical situation when  $\lambda = 2$ .

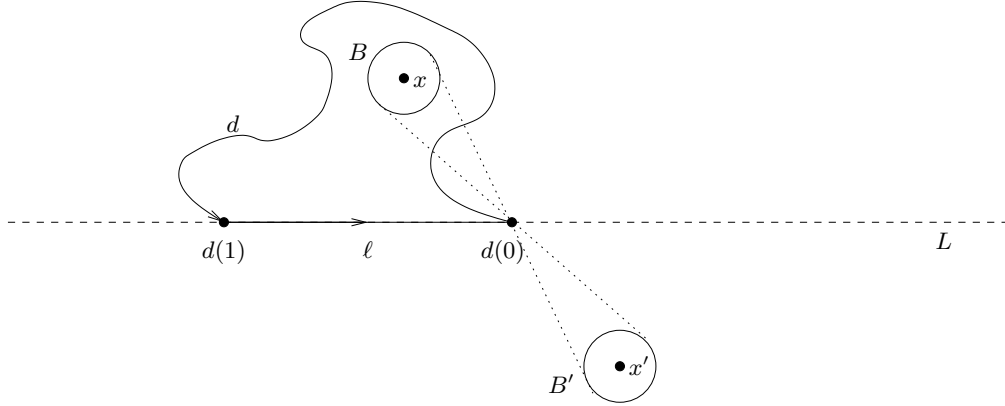


Figure 16: A typical path  $d$  satisfying the two properties of Lemma 93 when  $\lambda = 2$ . The loop  $\ell$  has nonzero winding number about every point in the ball  $B$  centered at  $x$ . The ball  $B'$  centered at  $x'$  is also shown.

We finish the proof by showing that  $B' \subseteq Q_\lambda(d)$ , whence  $Q_\lambda(d) = \mathbb{C}$  by Proposition 33. We do this in two steps: (i) we define a loop  $\ell'$  entirely included in  $Q_\lambda(d)$  that has nonzero winding number about every point  $y' \in B'$ ; and (ii) we exhibit a homotopy  $h$  from  $\ell'$  to the constant loop  $d(0)$  that stays entirely within  $Q_\lambda(d)$ . Now assuming we can do this, suppose there exists some  $y' \in B' - Q_\lambda(d)$ . Then since  $h$  avoids  $y'$ , it must keep the winding number about  $y'$  invariant throughout the deformation of the loop, but this is impossible, because the winding number of  $\ell'$  about  $y'$  is nonzero whereas the winding number of the constant loop  $d(0)$  about  $y'$  is zero. Thus no such  $y'$  can exist, and so  $B' \subseteq Q_\lambda(d)$  as desired.

The loop  $\ell'$  is made up of three segments: the first two are similar to  $d$ , and the third is  $d$  itself in reverse. We define  $\ell' : [0, 1] \rightarrow \mathbb{C}$  formally as follows: for all  $s \in [0, 1]$ ,

$$\ell'(s) := \begin{cases} \rho_{d(3s),d(0)}(\lambda) & \text{if } 0 \leq s \leq 1/3, \\ \rho_{d(1),d(3s-1)}(\lambda) & \text{if } 1/3 \leq s \leq 2/3, \\ d(3-3s) & \text{if } 2/3 \leq s \leq 1. \end{cases}$$

It is clear that  $\ell'$  stays entirely within the set  $Q_\lambda(d)$ , since it contains only  $\lambda$ -extrapolations of points on  $d$ . For convenience, we let  $a := \ell'(1/3) = \rho_{d(1),d(0)}(\lambda)$ . Note that  $a$  is colinear with  $d(0)$  and  $d(1)$ , since  $\lambda$  is real. Figure 17 shows the  $\ell'$  constructed from the path  $d$  of Figure 16.

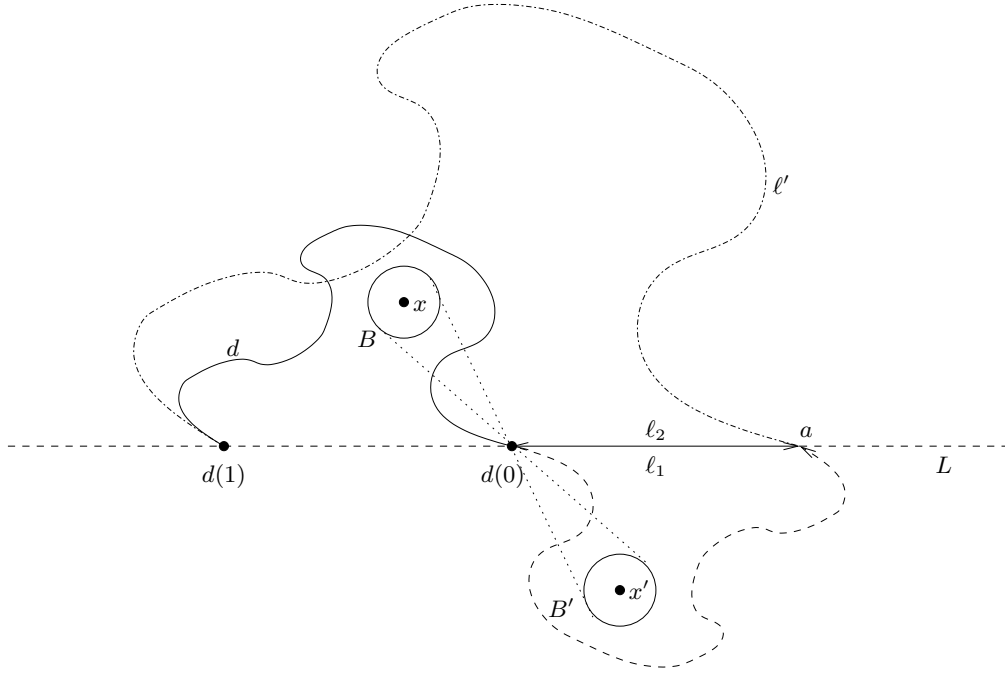


Figure 17: The loop  $\ell'$  constructed from the path  $d$  of Figure 16. The loop starts at  $d(0)$ , follows the dashed curve to the point  $a$ , then the dots and dashes to  $d(1)$ , then the curve  $d$  (solid) backwards to  $d(0)$ . The sets  $B$  and  $B'$  are also shown. The line segment between  $d(0)$  and  $a$  is used to split  $\ell'$  into the sum of two loops:  $\ell_1$  lying below  $L$  and  $\ell_2$  lying above  $L$ .



The loop  $\ell'$  is clearly homologous to the sum of two separate loops:  $\ell_1$  follows  $\ell'$  from  $d(0)$  to  $a$  (dashed curve in Figure 17) then a straight line segment along  $L$  from  $a$  back to  $d(0)$ ;  $\ell_2$  first goes straight from  $d(0)$  to  $a$  along  $L$ , then follows the second two thirds of  $\ell'$  from  $a$  back around through  $d(1)$  to  $d(0)$ . Notice that both  $B'$  and  $\ell_1$  are obtained by first rotating  $B$  and  $\ell$  respectively by  $\pi$  about the point  $d(0)$  followed by dilating about  $d(0)$  by a factor of  $\lambda - 1$ . Thus by similarity,  $\ell_1$  has the same nonzero winding number about every point in  $B'$  as  $\ell$  does about every point in  $B$ . Also notice that  $\ell_2$  lies entirely to the other side of  $L$  as  $B'$ , because the middle third of  $\ell'$  is just a dilation of  $d$  about  $d(1)$  by a factor of  $\lambda$ . It follows that  $\ell_2$  has zero winding number about every point in  $B'$ , and thus we conclude that  $\ell'$  has the same nonzero winding number around  $B'$  as  $\ell_1$  does.

Finally, we exhibit the promised homotopy  $h : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$  from  $\ell'$  to the constant loop  $d(0)$ : for all  $s, t \in [0, 1]$ , define

$$h(s, t) := \begin{cases} \rho_{d(3s(1-t)), d(0)}(\lambda) & \text{if } 0 \leq s \leq 1/3, \\ \rho_{d(1-t), d((3s-1)(1-t))}(\lambda) & \text{if } 1/3 \leq s \leq 2/3, \\ d((3-3s)(1-t)) & \text{if } 2/3 \leq s \leq 1. \end{cases}$$

It is easy to check that  $h$  is the desired homotopy, that is,  $h$  is continuous, and for all  $s, t \in [0, 1]$ , we have:  $h(s, 0) = \ell'(s)$ ;  $h(0, t) = h(1, t) = d(0)$ ;  $h(s, 1) = d(0)$ ; and  $h(s, t) \in Q_\lambda(d)$ .  $\square$

We need one more lemma before we prove Theorem 90. The preceding proof does not quite work as is when  $\lambda \notin \mathbb{R}$ , because the point  $x' = \rho_{x, d(0)}(\lambda)$  shown in Figure 16 may not lie below the line  $L$ , which means it may be tangled up with  $d$  in such a way that the winding numbers of the two loops  $\ell_1$  and  $\ell_2$  (see Figure 17) may cancel, leaving a zero winding number of  $\ell'$  about  $x'$  when we need it to be nonzero. To fix this, we do not use  $\lambda$  but instead use another point  $\mu \in Q_\lambda$  that is close enough to being real that the point  $\rho_{x, d(0)}(\mu)$  does lie below  $L$ . Then the whole proof of Proposition 94 goes through with  $\mu$  replacing  $\lambda$ .

For any  $z \neq 0$ , we define  $\arg z$  to be the unique  $\theta \in [0, \tau[$  such that  $z = |z|e^{i\theta}$ .

**Lemma 95.** *For any  $\lambda \in \mathbb{C} - [0, 1]$  and any  $\varepsilon > 0$ , there exists  $\mu \in Q_\lambda - \{1\}$  such that  $\arg(\mu - 1) < \varepsilon$ .*

*Proof.* Assume, without loss of generality, that  $\varepsilon < \pi/2$ . It suffices to find a point  $\mu \in Q_\lambda$  such that  $\Re(\mu) > 2$  and  $\arg \mu < \tan^{-1}((\tan \varepsilon)/2)$ . That is,  $\mu$  is somewhere in the shaded region in Figure 18.

We know that  $Q_\lambda$  is unbounded by Corollary 39. Fix some  $\nu \in Q_\lambda$  with  $|\nu| > 1$ , and note that all positive powers of  $\nu$  are in  $Q_\lambda$ . If  $(\arg \nu)/\tau$  is rational, then there exists  $n_0 \in \mathbb{Z}^+$  such that  $\arg(\nu^{kn_0}) = 0$  for all  $k \in \mathbb{Z}^+$ ; then pick  $k$  large enough so that  $\mu := \nu^{kn_0}$  has real part  $> 2$ . If  $(\arg \nu)/\tau$  is irrational, then a standard pigeonhole argument shows that the set  $\{(n \arg \nu) \bmod \tau \mid n \in \mathbb{Z}^+\}$  is dense in  $[0, \tau[$ , and so contains infinitely many points in  $[0, \varepsilon[$ . Thus we can find an  $n \in \mathbb{Z}^+$  such that  $\arg(\nu^n) = ((n \arg \nu) \bmod \tau) < \varepsilon$  and  $|\nu^n| = |\nu|^n$  is large enough to put  $\nu^n$  in the interior of the shaded region. Set  $\mu := \nu^n$ .  $\square$

Now we prove Theorem 90 by modifying the proof of Proposition 94 for nonreal  $\lambda$ .

*Proof of Theorem 90.* Let  $c$  be a bent path, and let  $\lambda$  be a point in  $\mathbb{C} - \mathbb{R}$ . As in the proof of Proposition 94, we can assume  $c$  includes no nonempty open subset of  $\mathbb{C}$  and replace  $c$  by a subpath  $d$  satisfying Lemma 93. As before, let  $x$  be given by that Lemma, let  $L$  be a straight line through  $d(0)$  and  $d(1)$  not containing  $x$ , and let  $\ell$  be the loop that follows  $d$  from  $d(0)$  to  $d(1)$

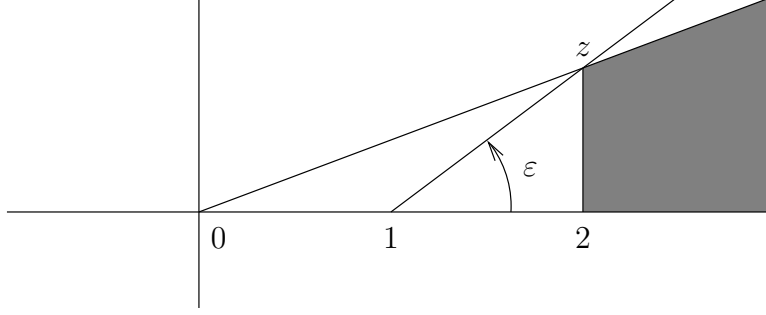


Figure 18: The shaded region is bounded by the real axis, the vertical line connecting 2 and  $z = 2 + i \tan \varepsilon$ , and the line from the origin through  $z$ . If  $\mu$  is in the closure of this region and  $\mu \neq z$ , then  $\arg(\mu - 1) < \varepsilon$ .

then back to  $d(0)$  along  $L$ . By extending  $d$  a little bit along  $L$  if necessary, we can assume that  $d(0) \neq d(1)$ . By reversing  $d$  if necessary, we can also assume that

$$\Im \left( \frac{x - d(1)}{d(0) - d(1)} \right) > 0,$$

that is, the three points  $d(1), d(0), x$  are oriented counterclockwise as they are in Figure 16.

Now let  $\varepsilon$  be the angle  $\angle x, d(0), d(1)$  formed by rays from  $d(0)$  through  $x$  and  $d(1)$ , respectively. That is,

$$\varepsilon = \arg \left( \frac{d(1) - d(0)}{x - d(0)} \right).$$

By our choice of orientation, we know that  $0 < \varepsilon < \pi$ . By Lemma 95, there exists a  $\mu \in Q_\lambda$  such that  $\arg(\mu - 1) < \varepsilon$ . By part (1.) of Lemma 21, we have  $Q_\mu(d) \subseteq Q_\lambda(d)$ , and so it suffices to show that  $Q_\mu(d)$  contains a nonempty open subset of  $\mathbb{C}$ . This will be just as we did in the proof of Proposition 94 but with the “almost real”  $\mu$  replacing the real  $\lambda$ . If  $\mu$  is in fact real, then  $\mu > 1$ , and we are done by Proposition 94, and so we assume  $\mu \notin \mathbb{R}$ , whence  $\Im(\mu) > 0$ . It follows that for any  $z, w \in \mathbb{C}$ , the three points  $z, w, \rho_{z,w}(\mu)$  are oriented counterclockwise.

Set  $x' := \rho_{x,d(0)}(\mu)$ . Then—and this the crucial point— $x'$  lies opposite the line  $L$  from  $x$ , as shown in Figure 19.

Figure 19 is analogous to Figures 16 and 17. As before, let  $a := \rho_{d(1),d(0)}(\mu)$ . Let  $L'$  be the line through  $d(0)$  and  $a$ , and let  $L''$  be the line through  $d(1)$  and  $a$  (see Figure 19). Notice that  $x'$  must be on the opposite side of  $L'$  from  $d(1)$ , and this together with the position of  $x'$  with respect to  $L$  implies that  $x'$  must be on the same side of  $L''$  as  $d(0)$ . As before, we can find an open ball  $B$  surrounding  $x$  such that: (i)  $\ell$  has the same nonzero winding number about every  $y \in B$  as it has about  $x$ ; and (ii) the open ball  $B' = \{\rho_{y,d(0)}(\mu) \mid y \in B\}$  surrounding  $x'$  intersects none of the three lines  $L, L'$ , or  $L''$ .

Now we define the loop  $\ell' \subseteq Q_\mu(d)$  similar to the proof of Proposition 94.

$$\ell'(s) := \begin{cases} \rho_{d(3s),d(0)}(\mu) & \text{if } 0 \leq s \leq 1/3, \\ \rho_{d(1),d(3s-1)}(\mu) & \text{if } 1/3 \leq s \leq 2/3, \\ d(3-3s) & \text{if } 2/3 \leq s \leq 1. \end{cases}$$

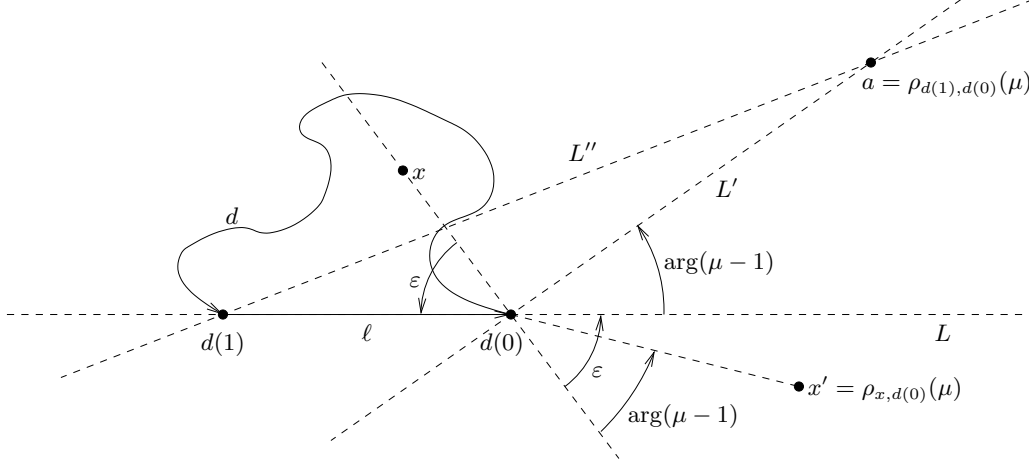


Figure 19: The point  $x' = \rho_{x,d(0)}(\mu)$  (lower right) lies below the line  $L$ , whereas  $x$  lies above  $L$ . The point  $a = \rho_{d(1),d(0)}(\mu)$  is also shown, as well as various lines and angles, including the line  $L'$  through  $d(0)$  and  $a$  and the line  $L''$  through  $d(1)$  and  $a$ . The point  $x'$  must also lie on the opposite side of  $L'$  as  $d(1)$ , and hence on the same side of  $L''$  as  $d(0)$ .

The first third of  $\ell'$  runs from  $d(0)$  to  $a$  and lies opposite  $L'$  from  $d(1)$ . As before, the loop  $\ell_1$  formed by closing this segment of  $\ell'$  along  $L'$  is a rotated, dilated copy of  $\ell$  and so has the same nonzero winding number about every point in  $B'$  as  $\ell$  does about  $x$ . The middle third of  $\ell'$  runs from  $a$  to  $d(1)$  and stays on the other side of  $L''$  from  $d(0)$ , and hence also from  $B'$ . Finally, our choice of  $\mu$  ensures that the last third of  $\ell'$ , which coincides with  $d$ , stays on the side of  $L$  opposite  $B'$ . Thus the loop  $\ell_2$  formed by closing off the final two thirds of  $\ell'$  along  $L'$  cannot contribute to the winding number of  $\ell'$  about any point in  $B'$ . It follows that  $\ell'$  has the same nonzero winding number about every point in  $B'$  as  $\ell_1$  has.

We define the homotopy  $h$  just as before, but with  $\mu$  instead of  $\lambda$ :

$$h(s, t) := \begin{cases} \rho_{d(3s(1-t)), d(0)}(\mu) & \text{if } 0 \leq s \leq 1/3, \\ \rho_{d(1-t), d((3s-1)(1-t))}(\mu) & \text{if } 1/3 \leq s \leq 2/3, \\ d((3-3s)(1-t)) & \text{if } 2/3 \leq s \leq 1. \end{cases}$$

This homotopy stays within  $Q_\mu(d)$  and contacts  $\ell'$  to the constant point  $d(0)$ , whose winding number about any point in  $B'$  is zero. Thus the curve must pass through each point in  $B'$  sometime during the deformation, and this puts  $B' \subseteq Q_\mu(d)$  as before. Hence  $\mathbb{C} = Q_\mu(d) = Q_\lambda(c)$ .  $\square$

## 14 Conjectures, open problems, and future research

We have many more questions than we can investigate in any reasonable length of time. We only give a sampling in this section. Some may be easy, but we have just not looked at them in depth.

Recall the set  $\mathcal{C}$  of Definition 17. We know that  $\mathcal{C}$  is open and contains all  $\lambda$  such that either  $0 < |\lambda| < 1$  or  $0 < |1 - \lambda| < 1$ . This is clearly not optimal, because we can use various constructions combined with Corollary 25 to carve out more territory for  $\mathbb{C}$  in the complex plane. This approach was started by Lemma 40 and Proposition 41. We still do not know much more about  $\mathcal{C}$ , however.

On the one hand, the only points we know not to be in  $\mathcal{C}$  are algebraic numbers (algebraic integers, in fact), and so it may be that  $\mathcal{C}$  contains all transcendental numbers. On the other hand, we currently have no proof that  $\mathcal{C}$  is unbounded; right now, we only know that  $\mathcal{C}$  contains at least some points with norm greater than 10. Further computer simulations will likely provide more points in  $\mathcal{C}$  with greater norm.

**Conjecture 96.**  $\mathcal{C}$  is unbounded.

**Conjecture 97.**  $\mathcal{C}$  contains all transcendental numbers.

**Conjecture 98.**  $\mathbb{C} - \mathcal{C}$  is a discrete set.

There are a number of open questions about  $R_\lambda$  when  $\lambda$  belongs to a discrete subring of  $D$ . For example, we conjecture that equality holds in Corollary 64.

**Conjecture 99.** The converse of Corollary 64 holds for all  $\lambda \in \mathbb{Z}$  such that  $\lambda \geq 2$ . That is,

$$R_\lambda = [\lambda(\lambda - 1)\mathbb{Z}] \cup [\lambda(\lambda - 1)\mathbb{Z} + 1] \cup [\lambda(\lambda - 1)\mathbb{Z} + \lambda] \cup [\lambda(\lambda - 1)\mathbb{Z} + 1 - \lambda]. \quad (18)$$

If  $\lambda$  lies in some discrete subring  $D \subseteq \mathbb{C}$ , then  $R_\lambda$  is also obviously discrete, because  $R_\lambda \subseteq D$ . We have examples, provided by Theorem 69, of  $\lambda$  such that  $R_\lambda$  is discrete for what must be nonobvious reasons, because  $\lambda$  is not an element of any discrete subring of  $\mathbb{C}$ , that is,  $\mathbb{Z}[\lambda]$  is not discrete. We will call such  $R_\lambda$  *essentially discrete*. All examples of essentially discrete  $R_\lambda$  we currently know about are consequences of Theorem 69.

**Open Question 100.** Are there any  $\lambda$  such that  $R_\lambda$  is discrete, but this fact is not a special case of Theorem 69?

The next open question is one of the most interesting.

**Open Question 101.** Does set equality always hold in (14)? Does it always hold in (15)? Does equality always hold in Corollary 80? If not, for what  $\lambda$  does it hold?

We know that equality holds for  $\lambda := -\varphi$  (equivalently,  $\lambda := 1 + \varphi$ ) by Proposition 48. We conjecture that it also holds for  $\lambda = -1 - \sqrt{2}$ , but we have not spent any time investigating this case.

**Research Plan 102.** Determine which  $\lambda > 3$  yield  $R_\lambda = \mathbb{R}$ . Determine which  $\lambda$  with  $\Re(\lambda) = 1/2$  yield  $R_\lambda = \mathbb{C}$ .

**Research Plan 103.** Get a reasonably good graphical picture of  $\mathcal{C}$ .

**Open Question 104.** How many elements of  $Q_{[x]}$  are there of degree 3?

The technique of Proposition 82 in Section 11 gives an upper bound of 717, and an extensive computer search finds only 90. Perhaps Fact 83 can help reduce the upper bound.

Call the triangle with vertices  $(0, 1, \lambda)$  the *fundamental triangle*. George McNulty offers the following conjecture:

**Conjecture 105** (McNulty). If  $R_\lambda$  contains a point in the interior of the fundamental triangle, then  $R_\lambda$  is convex.

Recall that  $O$  is a regular octagon.

**Conjecture 106.** *All  $\mathbb{R}$ -affine transformations of  $\mathbb{C}$  that are symmetries of  $R_{2+\sqrt{2}}(O)$  are length-preserving and leave  $O$  invariant.*

It is not too hard to see (via an induction on the length of a “construction”) that if  $S \subseteq \mathbb{C}$  is path-connected, then  $Q_\lambda(S)$  is path-connected for all  $\lambda$ .

**Open Question 107.** *If  $S \subseteq \mathbb{C}$  is path-connected, does that imply  $R_\lambda(S)$  is path-connected for all  $\lambda$ ? If  $T$  is  $\lambda$ -clonvex and path-connected, does that imply  $T$  is convex?*

**Definition 108.** Let  $L$  and  $S$  be any subsets of  $\mathbb{C}$ .

- $S$  is  $L$ -convex iff  $S$  is  $\lambda$ -convex for all  $\lambda \in L$ .
- A set  $S$  is  $L$ -clonvex iff  $S$  is  $L$ -convex and closed.
- Let  $Q_L(S)$  be the least  $L$ -convex superset of  $S$ ; let  $R_L(S)$  be the least  $L$ -clonvex superset of  $S$ .
- $S$  is *auto-clonvex* iff  $S$  is  $S$ -clonvex.

All the  $R_\lambda$  sets are auto-clonvex by Corollary 22. We conjecture the converse.

**Conjecture 109.** *If  $S \subseteq \mathbb{C}$  contains 0 and 1 and is auto-clonvex, then  $S = R_\lambda$  for some  $\lambda \in \mathbb{C}$ .*

This conjecture implies that the set  $\{R_\lambda \mid \lambda \in \mathbb{C}\}$  is closed under arbitrary intersections, because the intersection of any family of auto-clonvex sets is clearly auto-clonvex.

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